Spin-one ferromagnets with single-ion anisotropy in a perpendicular external field

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The conventional Holstein-Primakoff method is generalized with the help of the characteristic angle transformation [Lei Zhou and Ruibao Tao, J. Phys. A 27, 5599 (1994)] for spin-one magnetic systems with single-ion anisotropies. We find that the weakness of the conventional method for such systems can be overcome by this approach. Two models will be discussed to illuminate the main idea, which are the “easy-plane” and “easy-axis” spin-one ferromagnets, respectively. Comparisons show that the current approach can give reasonable ground-state properties for the magnetic system with easy-plane anisotropy, although the conventional method never can, and can give a better representation than the conventional one for a magnetic system with easy-axis anisotropy, although the latter is usually believed to be a good approximation in such a case. Study of the easy-plane model shows that there is a phase transition induced by the external field and the low-temperature specific heat may have a peak as the field reaches the critical value. [S0163-1829(96)04833-3]

I. INTRODUCTION

Magnetic systems with single-ion anisotropy $D(S_i^z)^2$ have been attracting attention for years since such a kind of anisotropy was found to be very popular in many magnetic materials. On the theoretical side, spin-wave excitations in such systems are not easy to handle, caused by the off-diagonal effect of the single-ion anisotropy, especially when the spontaneous magnetized direction is not the same as the anisotropic direction. Many theoretical approaches have been developed to deal with such kinds of systems. Usually, one first applies a rotating transformation of the spin vectors to determine the ground state and then performs a Holstein-Primakoff (HP) transformation to study the low-lying spin-wave excitations. However, this method was found to be a good approximation only when the anisotropy is the “easy-axis” case (i.e., $D<0$). In the “easy-plane” case (i.e., $D>0$), the method was much worse. To understand it, one can study an easy-plane Heisenberg model. If the conventional HP method is used naively to discuss the ground state and the magnon excitations of such a system, an imaginary value of the excitation energy for the “$k=0$” mode will always be encountered, which implies the failure of this method.

Such a deficiency of the conventional method is caused by missing an important quantum effect. Actually, for the single-ion anisotropy (no matter the easy-axis case or easy-plane case), an off-diagonal term $D \sin^2(\theta S_i^z)^2$ will always appear in the Hamiltonian as well as the diagonal terms $D \cos^2(\theta S_i^z)^2$ after introducing the spin vector rotation. Such an off-diagonal term may have the tendency to mix the single-site spin state $|n\rangle$ with $|n+2\rangle$ and $|n-2\rangle$ to form the proper eigenstates, and this spin-state mixing effect is completely a quantum one which is very important in the easy-plane anisotropy case. Unfortunately, such a quantum effect has been neglected by the conventional HP method. As a result, the conventional method has failed for magnetic systems with easy-plane anisotropy.

On the other hand, many methods have been proposed for easy-plane magnetic systems. The matching of the matrix elements (MME) method was one which can be used to consider the spin-state mixing effect perturbatively so that it can give a reasonable representation for an easy-plane ferromagnet when the single-ion anisotropy is small, and some numerical methods were developed for an easy-plane spin-one ferromagnet. Recently, another method—the characteristic angle (CA) method—was proposed for the easy-plane spin-one ferromagnet which could be applied to describe such spin-state mixing effects by a variation parameter through a spin operator transformation. The magnetic properties had been investigated for such a system in zero field, and the results seemed to be closer to the numerical results than those of the MME method.

The present work is focused on generalizing the conventional HP method with the help of a CA transformation for spin-one magnetic systems with single-ion anisotropies. Two particular models will be studied as the illustration of the CA approach, although the latter is certainly not limited to such models. The difficulties faced by the conventional HP method are overcome for such systems with our approach.

This paper is organized as follows. In the next section, the easy-plane model is studied using the CA approach. Detailed comparisons of the CA approach with the conventional method are made in Sec. III. Section IV is devoted to an easy-axis model, and the conclusions are summarized in the last section.

II. EASY-PLANE CASE

The first model we will study is an easy-plane spin-one ferromagnet in an external magnetic field which is applied
perpendicularly to the easy plane. The Hamiltonian of this system can be given as

\[
H = -J \sum_{\langle ij \rangle} S_i \cdot S_j + D \sum_i (S_i^z)^2 - h \sum_i S_i^z, \tag{1}
\]

where the first term is the exchange interaction, and the second one is the single-ion anisotropy. The anisotropy parameter \(D\) is positive so that the \(x\)-\(y\) plane is the so-called easy-plane and the \(z\) axis is the "hard axis." An external magnetic field \(h\) is applied along the hard axis.

Although the single-site part of the Hamiltonian, \(D(S_i^z)^2 - h S_i^z\), has already been expressed in diagonalized form, it is still unreasonable to apply a HP transformation naively to discuss the magnetic properties of such a system assuming the ground state to be the ordinary ferromagnetic state. Actually, if we do that, we will easily find that the magnon excitation energy of the \(k = 0\) mode will be always negative in the case of \("h < D"\). That is because the "starting point" based on which the spin deviations are discussed is wrong.

One must be very careful in finding a reasonable starting point. Actually, in such a system, on the one hand, the spins are forced into the easy-plane by the single-ion anisotropy; on the other hand, they have the tendency to point along the hard axis caused by the external field. As a result, these two effects must compete with each other and a new direction \(z^'\) axis would be optimized to describe the spontaneous magnetized direction. So it is desirable to introduce a new coordinate system \((\hat{x} ', \hat{y} ', \hat{z} ')\) in which the spin components are related to those in the original coordinates by the following transformation:

\[
S_i^z = \cos \theta_z S_i^z - \sin \theta_z S_i^z, \quad (2)
\]

\[
S_i^y = \cos \theta_z S_i^y + \sin \theta_z S_i^y, \quad (3)
\]

\[
S_i^x = S_i^z' . \quad (4)
\]

Applying the above transformation to Hamiltonian (1), we have

\[
H = -J \sum_{\langle ij \rangle} S_i^z' \cdot S_j^z' + D \cos^2 \theta_z \sum_i (S_i^z')^2 + D \sin^2 \theta_z \sum_i (S_i^z')^2
- h \cos \theta_z \sum S_i^z' - D \sin \theta_z \cos \theta_z \sum (S_i^z S_i^z + S_i^z S_i^z)
+ h \sin \theta_z \sum S_i^z . \quad (5)
\]

In the classical view, we can always determine \(\theta_z\) based on the variation method assuming that all spins are aligned along the \(z^'\) direction in the ground state. However, one should be careful in the quantum case, especially in the current easy-plane anisotropy case. Actually, if we apply the HP transformation naively to investigate the spin-wave excitation in such a system, an imaginary value of the magnon excitation energy for the \(k = 0\) mode will always exist. In fact, since

\[
D \sin^2 \theta_z (S_i^y')^2 = \frac{D}{4} \sin^2 \theta_z (S_i^y S_i^y + S_i^y S_i^y)
+ \frac{D}{4} \sin^2 \theta_z (S_i^z S_i^z + S_i^z S_i^z), \quad (6)
\]

if the HP transformation is applied naively to Hamiltonian (5), one may find that the off-diagonal terms \((D/4) \sin^2 \theta_z (S_i^y S_i^y + S_i^y S_i^y)\) in the above equation have no contribution to the constant term of the transformed Hamiltonian. That means the spin-state mixing effect has already been neglected by the conventional HP method. Unfortunately, such an effect is very important and must be considered in such a case. The characteristic angle (CA) transformation\(^\text{10}\) was developed to describe the spin-state mixing effect in the spin-one case by introducing another variation parameter \(\theta_c:\)

\[
S_i^z' = \cos \theta_z S_i^z + \sin \theta_z S_i^z \exp (i \pi \hat{S}_i^z), \quad (7)
\]

\[
S_i^y' = \cos \theta_c S_i^y + \sin \theta_c \exp (-i \pi S_i^y) S_i^y' , \quad (8)
\]

\[
S_i^x' = (1/2) [S_i^x S_i^x - S_i^x S_i^x]. \quad (9)
\]

The spin operators are transformed into a new set of quasispin operators \((\tilde{S}_i^x, \tilde{S}_i^y)\) which have been proved to obey all spin-one operator commutation rules.\(^\text{10}\) After the CA transformation, we can apply a HP transformation to transform the quasispin operator to a Bose one,

\[
\tilde{S}_i^z \to 1 - a_i^\dagger a_i, \quad (10)
\]

\[
\tilde{S}_i^y \to \sqrt{2} \sqrt{1 - (a_i^\dagger a_i/2)} a_i , \quad (11)
\]

\[
\tilde{S}_i^y \to \sqrt{2} a_i^\dagger \sqrt{1 - (a_i^\dagger a_i/2)} . \quad (12)
\]

Then, the Hamiltonian will have the form

\[
H = U_0 + H_1 + H_2 + \cdots, \quad (13)
\]

where

\[
U_0 = \frac{1}{2} \sum_{\langle ij \rangle} J (S_i S_j - S_i S_j) - D \cos^2 \theta_c (1 + \sin 2 \theta_c)
- h \cos \theta_c \cos 2 \theta_c, \quad (14)
\]

\[
H_1 = - \frac{\sqrt{2}}{2} \sum_i [D \sin \theta_c \cos \theta_c (\cos \theta_c + \sin \theta_c)
- h \sin \theta_c (\cos \theta_c - \sin \theta_c)] (a_i^\dagger + a_i) , \quad (15)
\]

and \(H_2\) can be written in moment in momentum \(k\) space as follows:

\[
H_2 = \sum_k A_k a_k^\dagger a_k + \sum_k B_k (a_k^\dagger a_k + a_k a_k) \quad (16)
\]

\[
A_k = 2JZ (\cos \theta_c - \gamma_k) + \frac{D}{2} \sin^2 \theta_c (1 + \sin 2 \theta_c)
+ h \cos \theta_c \cos 2 \theta_c, \quad (17)
\]
$$B_k = \frac{\sqrt{2}}{2} \left[ \frac{\gamma_k}{-JZ\sin 4\theta_c + \frac{D}{2}\sin^2\theta_c\cos 2\theta_c - h\cos\theta_c\sin 2\theta_c} \right]$$

$$+ JZ\sin 2\theta_c \gamma_k. \quad (18)$$

Based on the variation method we understand that the two parameters \(\theta_r\) and \(\theta_c\) should be determined by minimizing the ground-state energy. As a first-order approximation, we may obtain

$$\frac{1}{N} \frac{d}{d\theta_r} U_0(\theta_r, \theta_c) = -D\sin\theta_r\cos\theta_r(1 + \sin 2\theta_c)$$

$$+ h\sin\theta_r\cos 2\theta_c = 0, \quad (19)$$

$$\frac{1}{N} \frac{d}{d\theta_c} U_0(\theta_r, \theta_c) = 4JZ\sin 2\theta_c\cos 2\theta_c - D\sin^2\theta_c\cos 2\theta_c$$

$$+ 2h\cos\theta_r\sin 2\theta_c = 0. \quad (20)$$

Equation (19) is just the same as the condition \(H_1 = 0\) and Eq. (20) can cancel most of the off-diagonal terms which are in the square brackets in the expression of \(B_k\). If we substitute the solution of the above nonlinear equations into Hamiltonian (13) and then diagonalize the harmonic part of Hamiltonian \(H_2\) by the usual Bogolyubov transformation, the total Hamiltonian will be

$$H = U_0 + \sum_k E_k \alpha_k^\dagger \alpha_k + \cdots, \quad (21)$$

where

$$U_0' = U_0 - \frac{1}{2} \sum_k A_k + \frac{1}{2} \sum_k \sqrt{A_k^2 - 4B_k^2}, \quad (22)$$

$$E_k = \sqrt{A_k^2 - 4B_k^2}. \quad (23)$$

The ground state in such a method can be defined by

$$\alpha_0|0\rangle = 0. \quad (24)$$

Then the induced magnetization \(M(h)\) is derived in the harmonic approximation as follows:

$$M(h) = \frac{1}{N} \sum_i \langle 0 | S_i^z | 0 \rangle = \frac{1}{N} \sum_i \cos\theta_r \cos\theta_c \langle 0 | S_i^z | 0 \rangle = \frac{1}{N} \sum_i \cos\theta_r \cos\theta_c \langle 0 | S_i^z | 0 \rangle$$

$$\langle 0 | S_i^z | 0 \rangle = \cos\theta_r \cos\theta_c \langle 0 | S_i^z | 0 \rangle + \sin\theta_r \cos\theta_c \langle 0 | S_i^z | 0 \rangle$$

$$= \frac{1}{N} \sum_i \cos\theta_r \cos\theta_c \langle 0 | S_i^z | 0 \rangle + \frac{1}{N} \sum_i \sin\theta_r \cos\theta_c \langle 0 | S_i^z | 0 \rangle$$

$$= \frac{3}{2} \cos\theta_r \cos\theta_c - \frac{1}{2N} \sum_i \cos\theta_r \cos\theta_c \langle 0 | S_i^z | 0 \rangle$$

$$= \frac{\sqrt{2}}{2} \cos\theta_r \cos\theta_c \langle 0 | S_i^z | 0 \rangle = \frac{\sqrt{2}}{2} \cos\theta_r \cos\theta_c \langle 0 | S_i^z | 0 \rangle$$

Thus, putting the solution of Eqs. (19) and (20) into Eqs. (22), (23), and (25), such physical properties as the ground-state energy, the magnon dispersion relation, and the induced magnetization can be obtained. However, since it is very difficult to solve the nonlinear equations analytically, numerical calculations are carried out. The system with anisotropy parameter \(D/AJZ = 0.6\) has been studied as an example. \(\theta_r\) and \(\theta_c\) as functions of the external field have been drawn together in Fig. 1, from which one can find that they are both the decreasing function of the external field. It is understood that \(\theta_r\) is used to describe the spontaneous magnetized direction and \(\theta_c\) the spin-state mixing effect; in the zero applied field case, the spontaneous magnetized direction will be the \(x\) axis (\(\theta_r = 90^\circ\)), and the spin-state mixing effect should be the strongest since the off-diagonal term \(D\sin^2\theta_c(S_i^z)\) is the strongest and the value of \(\theta_c\) is consistent with Ref. 10 where the \(h = 0\) case has already been discussed. While the external magnetic field is strengthened, the spins will point along a direction which is closer to the \(z\) axis due to the interaction with the external field so that \(\theta_r\) will decrease; at the same time, the spin-state mixing effect is also weakened since the off-diagonal interactions in the total Hamiltonian will turn smaller along with the decrement of \(\theta_r\). However, when \(h\) reaches a critical value \(h_c = D\), the external magnetic field is so strong that the spins will not be rotated any longer and the off-diagonal term will come to zero; as a result, \(\theta_r\) and \(\theta_c\) will vanish simultaneously.

**III. COMPARISONS AND DISCUSSION**

In this section, we will compare the CA method with the conventional HP method in detail and discuss the magnetic properties of the above-mentioned system.

First, one may find that more quantum effects have been included in the constant term of the Hamiltonian by our approach.

Introducing the HP transformation naively to Hamiltonian (5), the constant term can be found as

$$U_0^{HP} = N(-JZ + D\cos^2\theta_r - h\cos\theta_r) + N \frac{D}{2}\sin^2\theta_r$$

$$= U_0^C + N \frac{D}{2}\sin^2\theta_r, \quad (26)$$

where \(U_0^C\) is the ground-state energy obtained by a classical rotating transformation.

After applying the CA transformation, the ground-state energy is Eq. (14) which can be rewritten as
where

\[ U_1 = N \left( J_Z \sin^2 \theta_c \sin \theta_r \cos \theta_r + h \cos \theta_r \sin^2 \theta_r \right) \]

\[ \theta_c \]

\[ U_0 = U_0^{\text{HP}} + U_1, \]

(27)

\[ U_1 \] is an additional term introduced by the CA transformation which will vanish as \( \theta_c = 0 \).

The conventional HP method is a semiclassical one. The only quantum effect in Eq. (26) is the term \( N(D/2) \sin^2 \theta_r \) which comes from the contribution of \( (D/4) \sin^2 \theta_r \left( S_i^x S_i^x + S_i^y S_i^y \right) \) in Eq. (6), and other terms which have been collected in \( U_0^{\text{HP}} \) of Eq. (26) can be easily recovered by a classical method. However, after applying the CA transformation, it can be clearly found that there is an additional contribution \( U_1 \) in the expression of \( U_0 \) which describes the single-site spin-state mixing effect through the variation parameter \( \theta_r \). Such an effect is completely a quantum one which has no classical counterpart, and it is expressed as the competition of the exchange term (\( J_Z \)) and the external term (\( h \)) with the single-ion anisotropy term (\( D \)).

So more quantum effects are considered by the CA approach than the conventional HP method even in the constant term of the Hamiltonian. Furthermore, considering such quantum effects will lead to a lower ground-state energy since the ground state in the CA method is selected to be the minimum point of the function \( U_0 \) although that in the conventional HP method is not so.

Now, one can compare the CA method with the conventional HP method for an elementary excitation of such a system. Putting the solution of \( \theta_r \) and \( \theta_c \) into Eq. (23), the magnon excitation gap can be calculated with respect to the external magnetic field and the result is shown in Fig. 2, where one can find that the magnon excitation gap obtained by the CA method will always be positive or zero.

However, based on the conventional HP method, one should obtain the variation parameter \( \theta_r \) by minimizing Eq. (26) which yields

\[ \frac{1}{N} \frac{d}{d \theta_r} U_0^{\text{HP}} = - D \sin \theta_r \cos \theta_r + h \sin \theta_r = 0; \]

(29)

then substituting the solution of the above equation into Eqs. (17), (18), and (23), one can easily find that the magnon excitation gap in the HP method will be

\[ \Delta^{\text{HP}} = \sqrt{h^2 - D^2}. \]

(30)

Of course, the excitation gap will never be real when \( h < D \). That is to say the conventional HP method cannot be applied naively to study magnetic systems with easy-plane anisotropy. However, the CA method has overcome this difficulty as shown in Fig. 2.

The induced magnetization as the function of the external magnetic field has been drawn in Fig. 3. From Figs. 1–3 one may find that the point \( h_c = D \) is very strange and there seems to be a phase transition at such a point. As the external field is strengthened across \( h_c \), the system transits to a phase in which the spins are not rotated any longer. Actually, this phase transition can be clearly shown by calculating the low-temperature specific heat \( C_v \).

FIG. 1. \( \theta_r \) and \( \theta_c \) as the functions of the external field for the easy-plane spin-one ferromagnet with \( D/4J_Z = 0.6 \).

FIG. 2. Magnon excitation gap as the function of the external field for the easy-plane spin-one ferromagnet with \( D/4J_Z = 0.6 \).

FIG. 3. Induced magnetization as the function of the external field for the easy-plane spin-one ferromagnet with \( D/4J_Z = 0.6 \).
Suppose the system is at low temperature, and only the low-energy excitation is considered; then the inner energy will be

\[ E(T) = E_0 + \sum_k E_k \exp(E_k/K_B T) - 1. \]

(31)

So the specific heat can be obtained:

\[ C_v = \frac{dE(T)}{dT} = \left[ \frac{1}{N} \sum_k (E_k/K_B T)^2 \exp(E_k/K_B T) \right] N K_B. \]

(32)

The specific heat of the system as the function of the external field at the temperature \( K_B T/JZ = 0.1 \) is shown in Fig. 4, in which a peak can be apparently found at the critical point \( h_c = D \). The physics can be understood as follows: In the vicinity of the phase transition point \( h_c \), the magnons will be excited without a gap (Fig. 2) so that the thermal fluctuations are strong.

### IV. EASY-AXIS CASE

Now we will study another model which is an easy-axis spin-one ferromagnet in an external magnetic field whose direction is perpendicular to the easy-axis. The Hamiltonian of such system can be given as

\[ H = -J \sum_{\langle i,j \rangle} S_i^z S_j^z - D \sum_i (S_i^z)^2 - h \sum_i S_i^z, \]

(33)

where the single-ion anisotropy makes the \( x \) axis an easy axis, and an external magnetic field is applied along the \( z \) direction.

At first glance, this Hamiltonian has the same classical picture as the last model—the anisotropic interaction and the interaction with external field have to compete with each other and will be balanced at some angle \( \theta_c \). So a rotating transformation of the spin vectors is helpful. In fact, many authors have used this method to discuss various kinds of magnetic systems with single-ion anisotropy and have believed that this approximation will work well when the

\[ \sin^2 \theta \sum_i (S_i^z)^2 - D \cos^2 \theta \sum_i (S_i^z)^2 \]

\[ -h \cos \theta \sum_i S_i^z \]

\[ + h \sin \theta \sum_i S_i^z. \]

(34)

There are still off-diagonal terms \( -D \cos^2 \theta \sum_i (S_i^z)^2 \), in the Hamiltonian, and these off-diagonal interactions may be important in some cases. So it is helpful to apply the CA transformation to get a more reasonable representation.

Actually, after almost the same procedure as that for the easy-plane model, the Hamiltonian can be transformed to

\[ H = U_0 + H_1 + H_2 + \cdots, \]

(35)

where

\[ U_0 = N \left[ -JZ \cos^2 \theta_c - D + \frac{D}{2} \cos^2 \theta_c (1 + \sin 2 \theta_c) \right], \]

(36)

\[ H_1 = -\frac{\sqrt{2}}{2} \sum_i [ -D \sin \theta_c \cos \theta_c \sin \theta_c ] (a_i^\dagger + a_i), \]

(37)

and

\[ H_2 = \sum_k A_k a_k^\dagger a_k + \sum_k B_k (a_k^\dagger a_k + a_k a_k^\dagger), \]

(38)

\[ A_k = 2JZ (\cos^2 \theta_c - \gamma_k) - \frac{D}{2} \cos^2 \theta_c (1 + \sin 2 \theta_c) \]

\[ + h \cos \theta_c \cos 2 \theta_c + D \sin^2 \theta_c, \]

\[ B_k = \frac{\sqrt{2}}{2} \left[ JZ \sin 4 \theta_c - \frac{D}{2} \cos^2 \theta_c \cos 2 \theta_c - h \cos \theta_c \sin 2 \theta_c \right] \]

\[ + JZ \sin 2 \theta_c \gamma_k. \]

(40)

The two variational parameters \( \theta_c \) and \( \gamma_k \) satisfy

\[ \frac{1}{N} \frac{d}{d\theta_c} U_0 (\theta_c, \theta_c) = -D \sin \theta_c \cos \theta_c (1 + \sin 2 \theta_c) \]

\[ + h \sin \theta_c \cos 2 \theta_c = 0, \]

(41)

\[ \frac{1}{N} \frac{d}{d\theta_c} U_0 (\theta_c, \theta_c) = 4JZ \sin 2 \theta_c \cos 2 \theta_c + D \cos^2 \theta_c \cos 2 \theta_c \]

\[ + 2h \cos \theta_c \sin 2 \theta_c = 0 \]

(42)

where Eq. (41) will cancel the \( H_1 \) part of the Hamiltonian.

Physical properties such as the magnon excitation and the induced magnetization have the same forms as in the last
model [i.e., Eqs. (22), (23), and (25)] except the concrete expression of the functions \( A_k \) and \( B_k \).

Very similar to the easy-plane case, the constant term in the Hamiltonian can further be divided into two terms:

\[
U_0 = U_0^{\text{HP}} + U_1,
\]

where

\[
U_0^{\text{HP}} = N(-JZ - D \sin^2 \theta_r - h \cos \theta_r) - N \frac{D}{2} \cos^2 \theta_r,
\]

\[
= U_0^c - N \frac{D}{2} \cos^2 \theta_r,
\]

\[
U_1 = N \left( JZ \sin^2 2 \theta_c + \frac{D}{2} \cos^2 \theta_r \sin^2 \theta_c + h \cos \theta_r \sin^2 \theta_r \right).
\]

\( U_0^{\text{HP}} \) is the contribution of the conventional HP method, and \( U_1 \) is an additional term which describes the quantum effect of spin-state mixing in a single site. The discussion is similar to the first model: More quantum effects have been included in the constant term of the Hamiltonian by the CA method, and considering such an effect in the CA method will lead to a lower ground-state energy than that in the conventional HP method. Actually, as shown in Fig. 5 where \( U_0 \) and \( U_0^{\text{HP}} \) are drawn together with respect to the external magnetic field for an easy-axis spin-one ferromagnet, \( U_0 \) is always found to be lower than \( U_0^{\text{HP}} \).

Now it is interesting to compare the elementary excitations calculated by the CA method with those by the conventional HP method. In the latter case, \( \theta_c = 0 \) and \( \theta_r \) is obtained by minimizing \( U_0^{\text{HP}} \) (44) which yields

\[
\frac{1}{\hbar} \frac{d}{d \theta_r} U_0^{\text{HP}} = -D \sin \theta_r \cos \theta_r + h \sin \theta_r = 0.
\]

So substituting the solution of the above equation back into Eqs. (39) and (40) and then into Eq. (23), the elementary excitation in the conventional HP method can be calculated readily.

The magnon excitation gaps have been calculated by both methods with respect to the external magnetic field, and the results are presented in Fig. 6 where the solid line is by the CA method and the dotted line is by the conventional HP method. From the figure one may find that when the external field is close to the anisotropy parameter \( D \), there is a small region where the magnon excitation gap calculated by the HP method will be imaginary, which indicates that this approximation is poor in such an area. However, the solid line in the figure tells us that CA method has overcome this difficulty and the magnon excitation gap will always be real and positive in the CA method. Actually, as shown in Fig. 7 where the values of the two variation parameters \( \theta_c \) and \( \theta_r \) are drawn together with respect to the external field, when \( h \) is close to \( D \), \( \theta_r \) comes to zero and \( \theta_c \) becomes somewhat larger, indicating that the spin-state mixing effect caused by the off-diagonal interaction may be very strong. So we must
consider such an effect with the help of the CA transformation in such a case; otherwise, the starting point may be unreasonable and will lead to an imaginary minimum excitation energy. Outside this region, the off-diagonal terms are not so strong compared to the diagonal parts; as a result, the spin-state mixing effect is not very drastic and the conventional HP method might be a reasonable approximation as many authors have believed. However, the CA transformation may always be helpful to get a more reasonable representation for such a system.

V. CONCLUSIONS

To summarize, in this paper, the conventional method has been generalized with the help of the characteristic angle transformation for spin-one magnetic systems. The difficulties faced by the conventional HP method for magnetic systems with single-ion anisotropy have been overcome by our approach. Two models have been discussed to illuminate the main ideas, of which one is an easy-plane spin-one ferromagnet in an external field applied perpendicular to the easy plane, and the other is an easy-axis spin-one ferromagnet in an external field applied perpendicularly to the easy axis. Comparisons between our approach and the old one show that more quantum effects have been considered by the CA method; as a result, the CA method can examine the ground-state properties of the easy-plane spin-one ferromagnet, although the old method never can, and the CA method can give an improved representation for the easy-axis spin-one ferromagnet, although the conventional HP method is usually believed to be a good approximation in such a case. Also, study of the easy-plane model shows that a phase transition may take place induced by the applied field, and the low-temperature specific heat is found to have a peak when the external field reaches the critical value.

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