Prepared by Jiang Xiao

第七章

刚体力学

什么是刚体

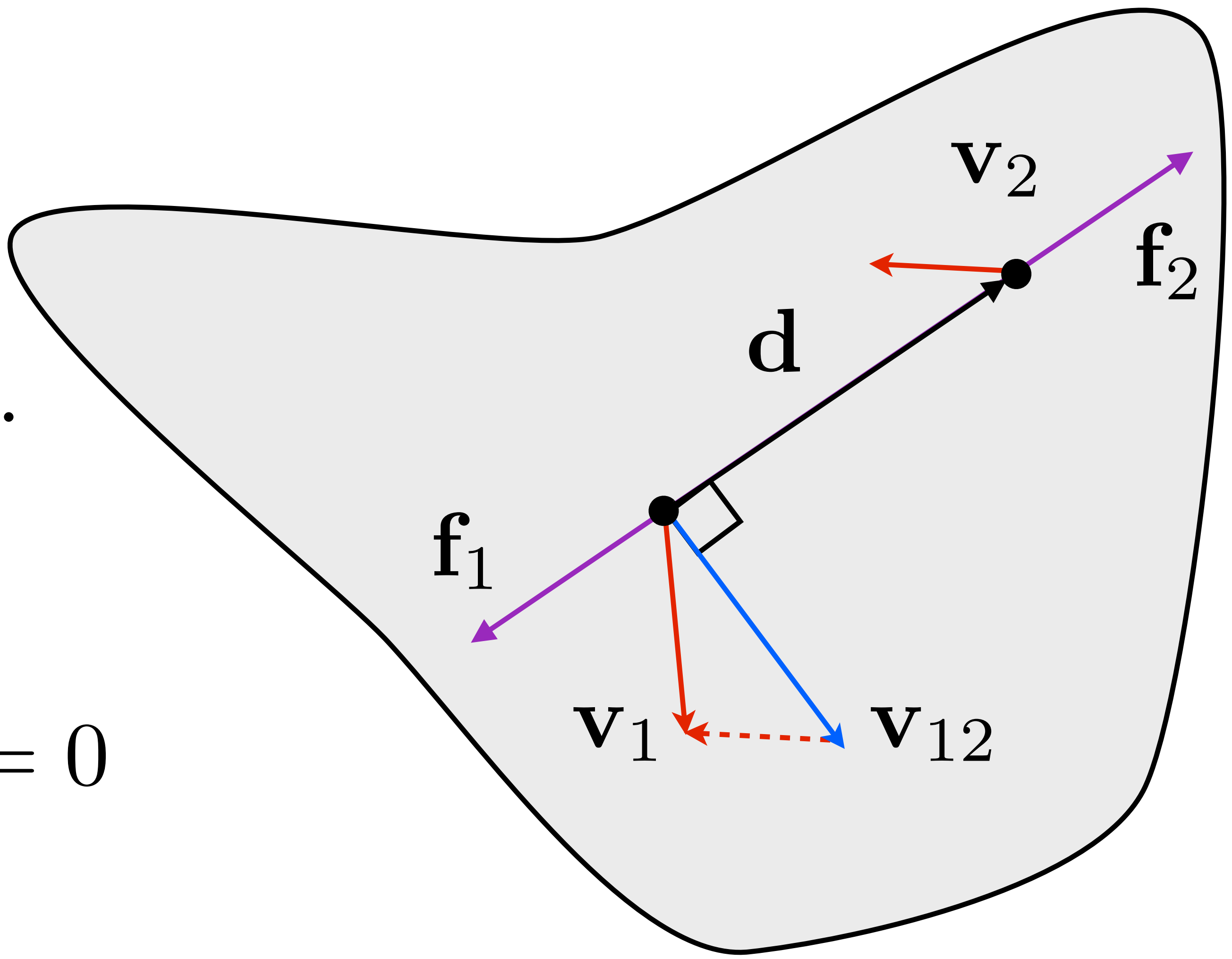
刚体不发生形变：

$$\mathbf{v}_{12} \perp \mathbf{d} \quad \Rightarrow \quad \mathbf{d} = \text{const.}$$

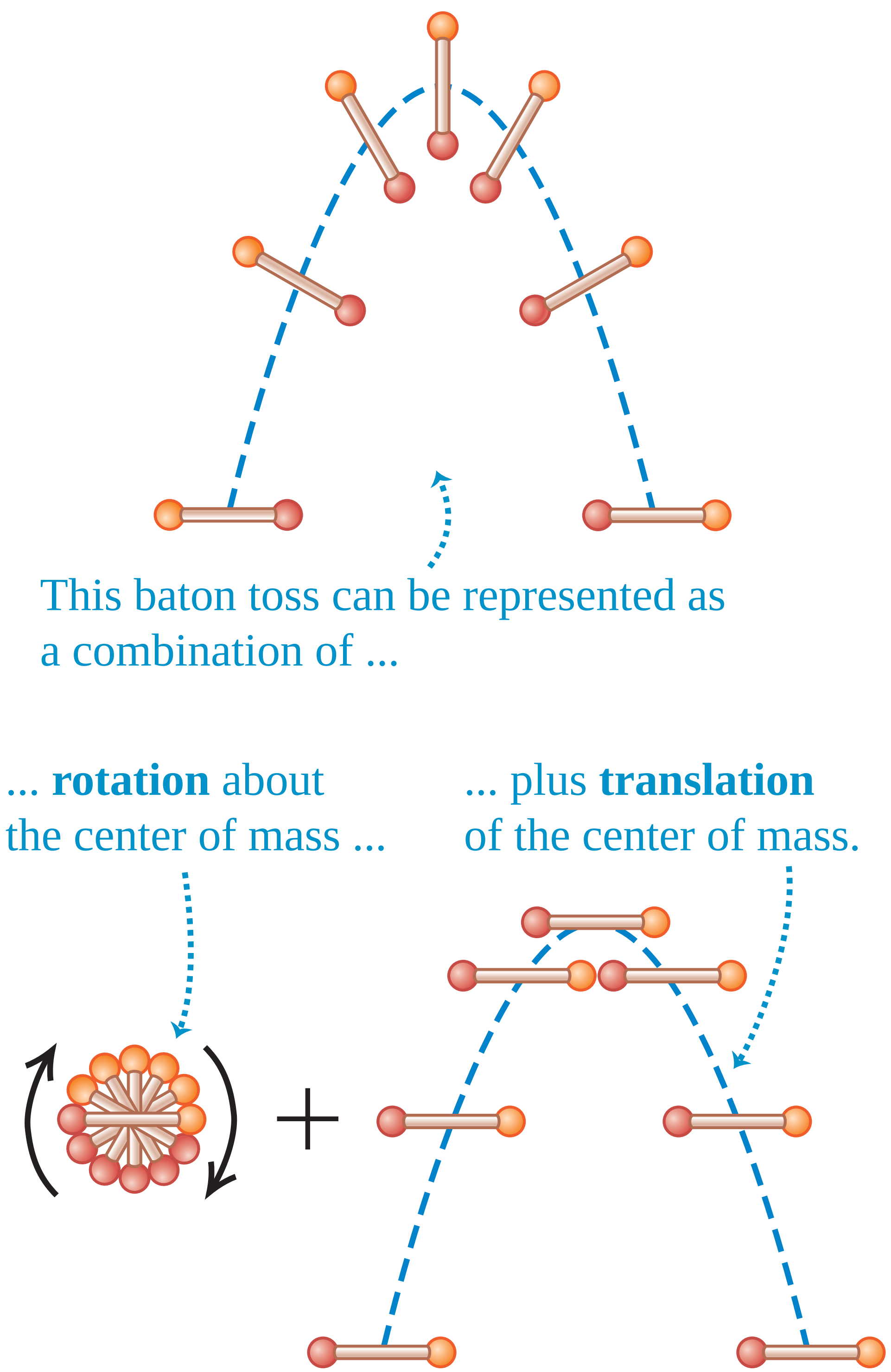
刚体内力不做功：

$$\mathbf{f}_1 \cdot \mathbf{v}_1 + \mathbf{f}_2 \cdot \mathbf{v}_2 = \mathbf{f}_1 \cdot \mathbf{v}_{12} = 0$$

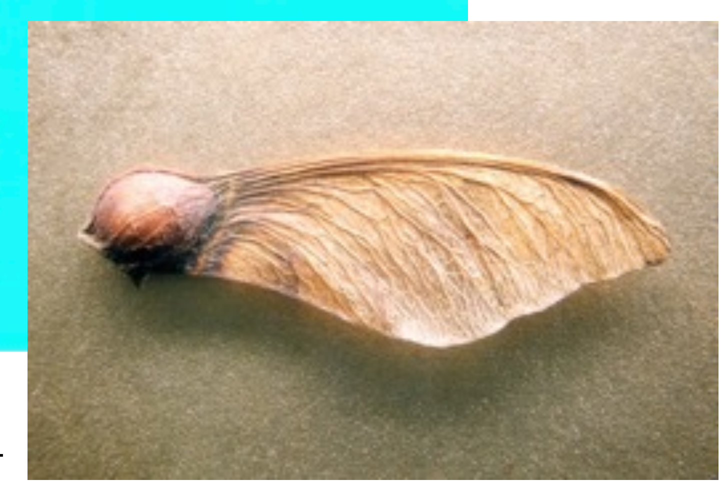
力在刚体中传播是瞬时的



平动和转动

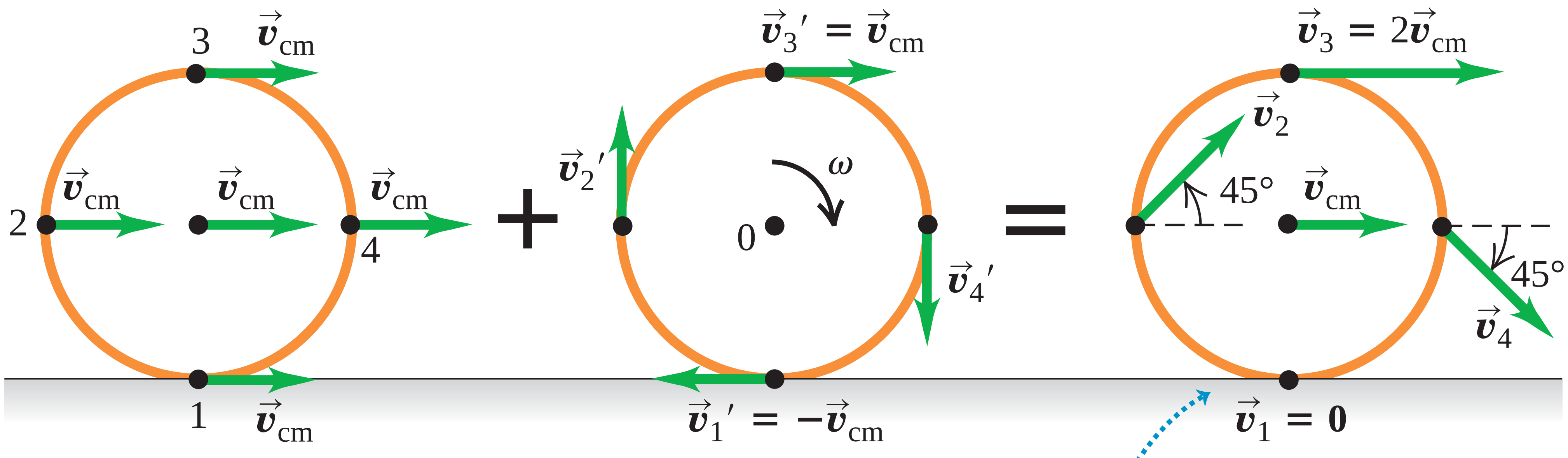


翅果



http://bcove.me/h99ffg3u

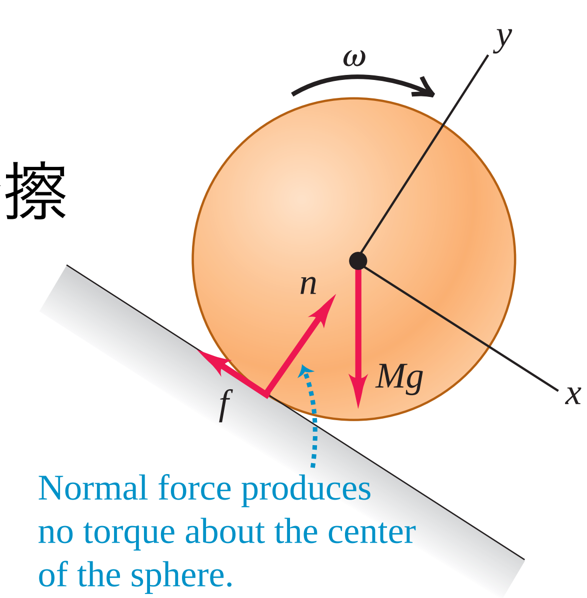
滚动

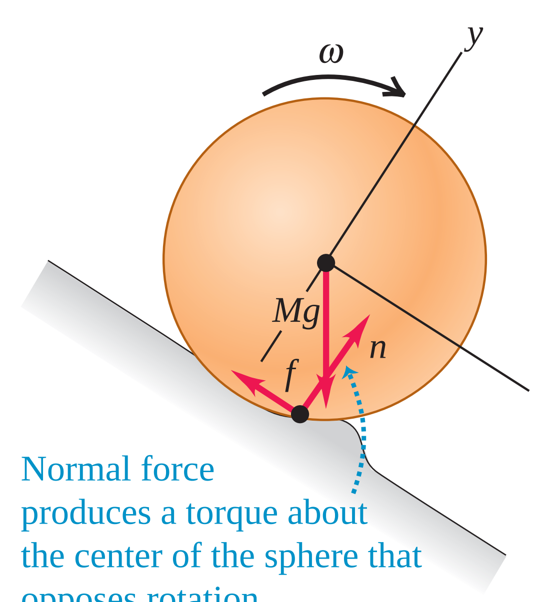


纯转动：$v_{\rm cm} = 0$　　纯滚动：$\omega r = v_{\rm cm}$

纯滑动：$\omega = 0$　　带滑动的滚动：$\omega r \neq v_{\rm cm}$

滑动摩擦



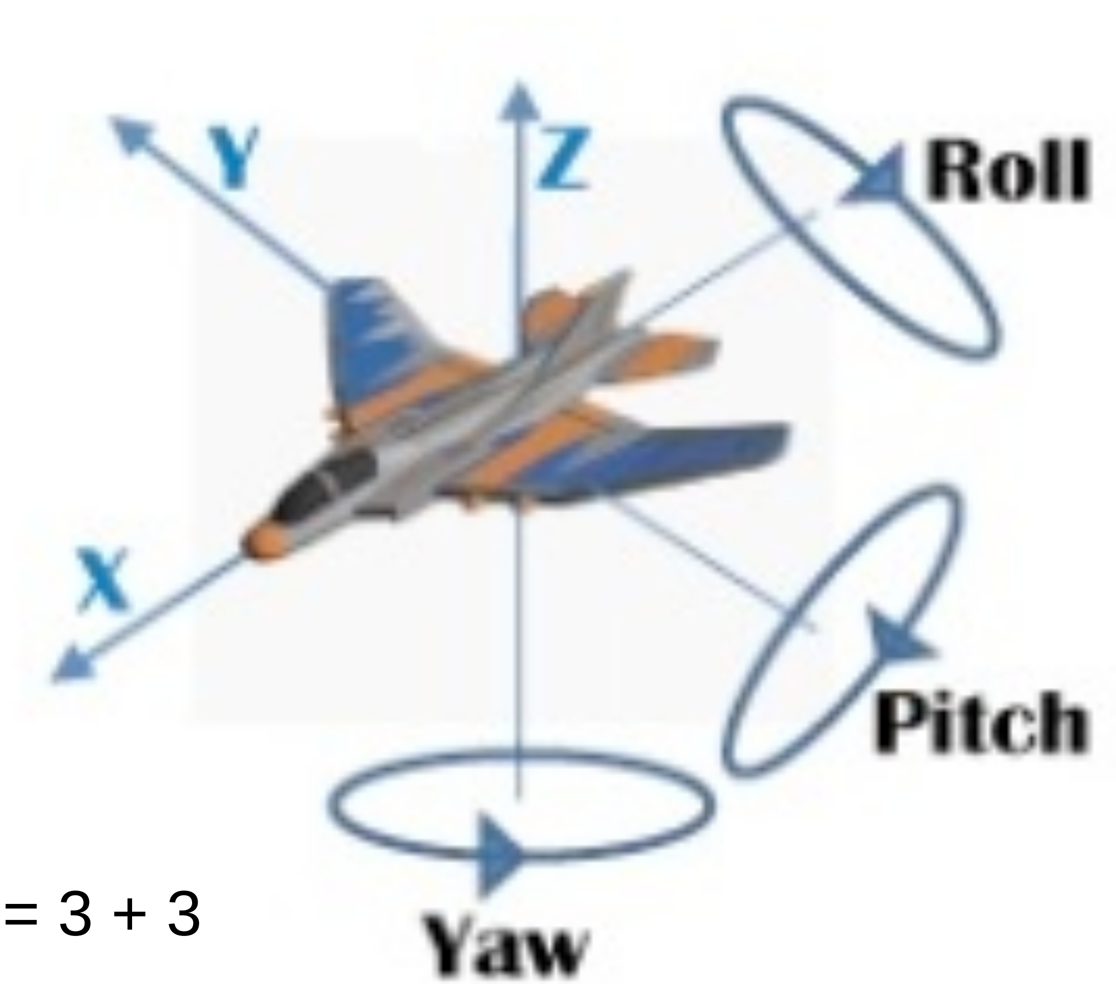


滚动摩擦

自由度

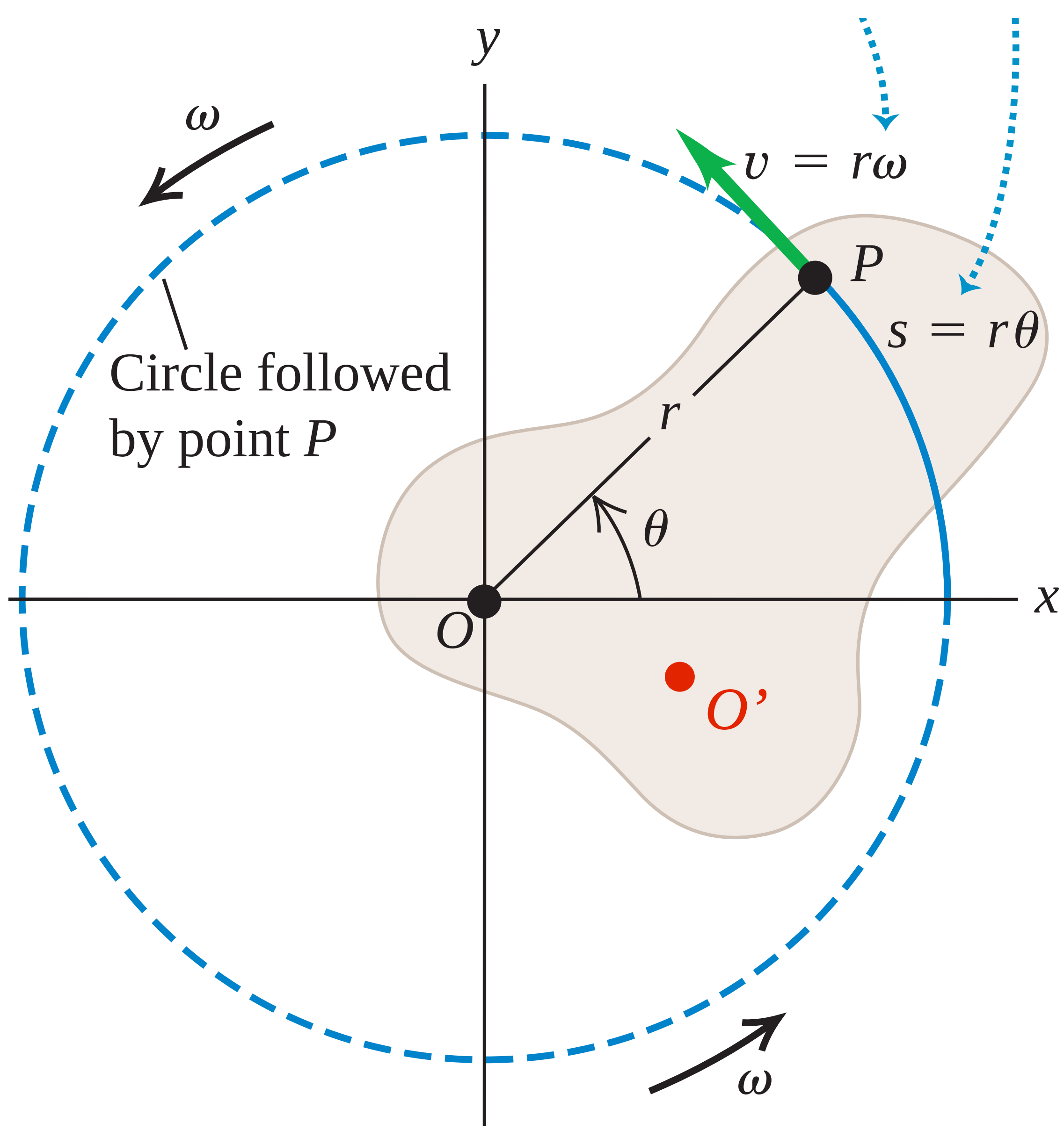
自由度：决定一物体的位置所需的独立坐标数。

刚体：总自由度 = 平动自由度 + 转动自由度



自由度 = 3 + 3

角速度和线速度



$$\mathbf{v} = \boldsymbol{\omega} \times \mathbf{r}$$

$$\begin{aligned}\frac{d\mathbf{v}}{dt} &= \frac{d(\boldsymbol{\omega} \times \mathbf{r})}{dt} \\ &= \frac{d\boldsymbol{\omega}}{dt} \times \mathbf{r} + \boldsymbol{\omega} \times \mathbf{v} \\ &= \frac{d\boldsymbol{\omega}}{dt} \times \mathbf{r} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r})\end{aligned}$$

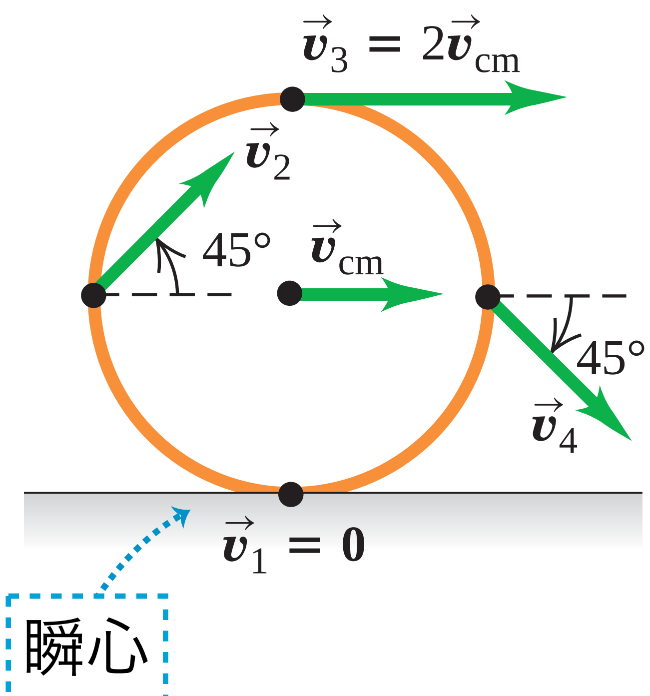
证明：P点绕O点和O'点的角速度相同。

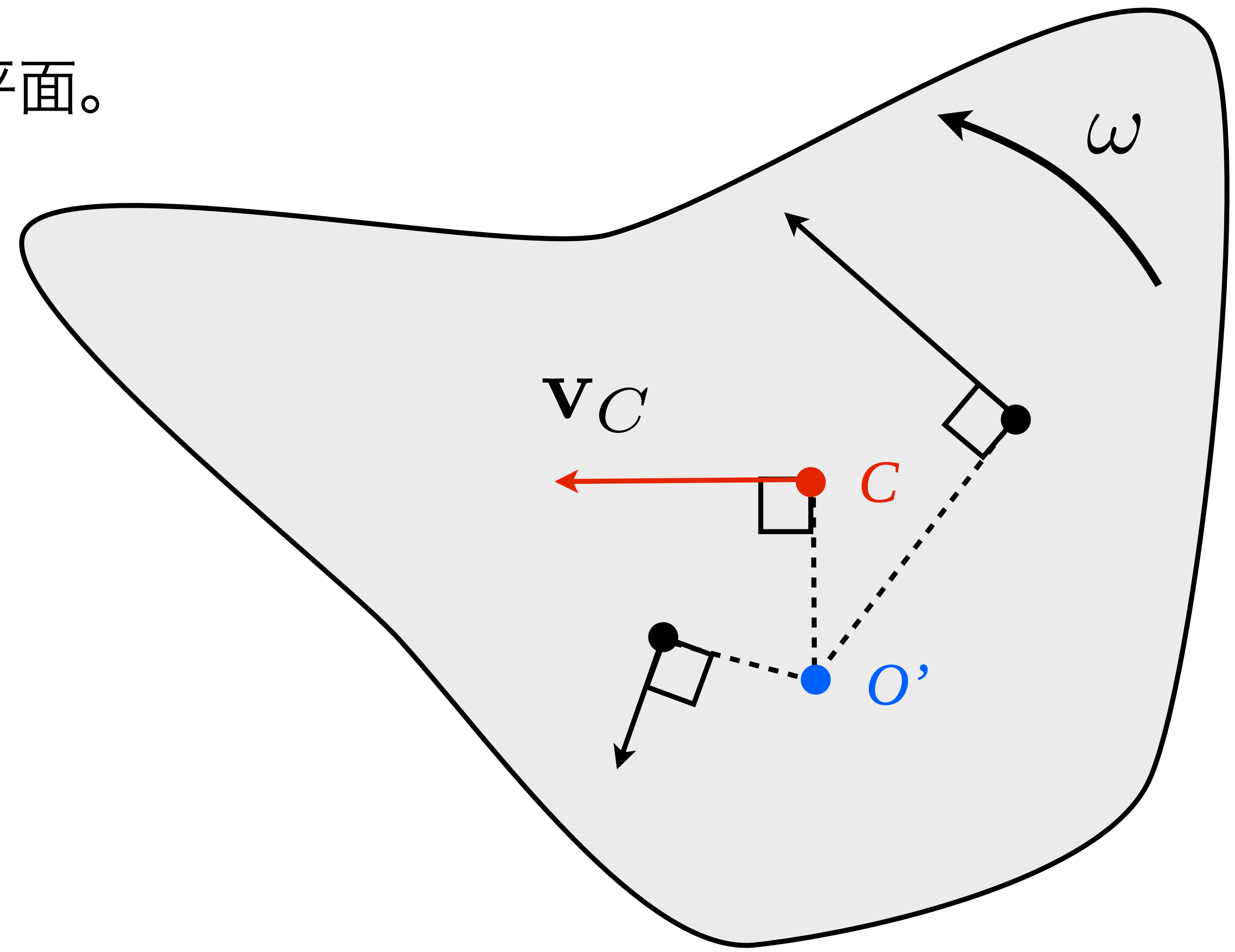
平面运动和瞬心

平面运动：所有点的运动都平行于某一平面。

速度： $\mathbf{v} = \mathbf{v}_C + \boldsymbol{\omega} \times \mathbf{r}$

垂直于运动平面

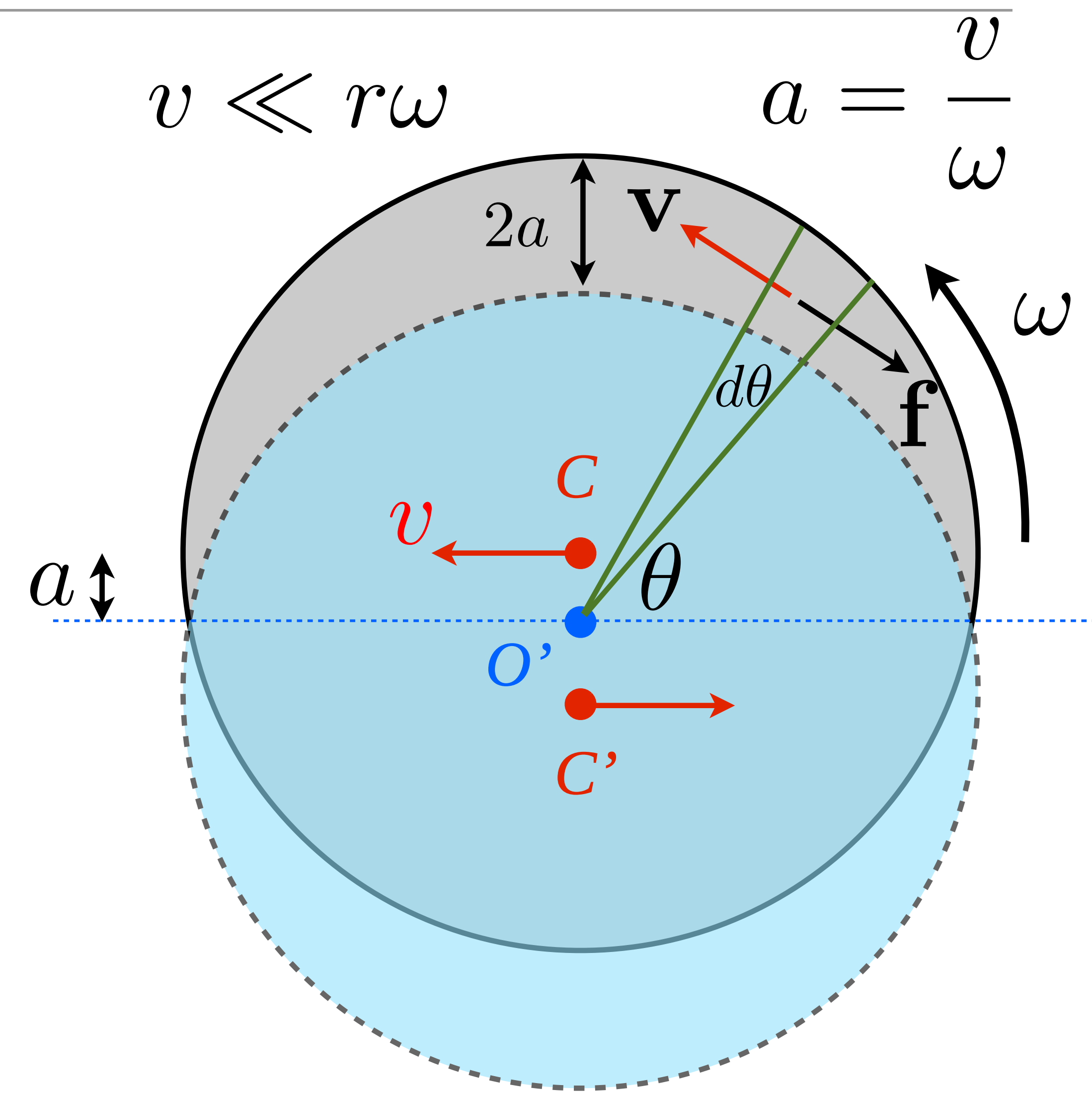




瞬心：瞬时速度为零的点

$$\mathbf{v}_C + \boldsymbol{\omega} \times \mathbf{r}_0 = 0$$

平面运动和瞬心


$v \ll r\omega$
$a = \dfrac{v}{\omega}$
$2a$
$\mathbf{v}$
ω
$d\theta$
$\mathbf{f}$
C
v
θ
a
O'
C'

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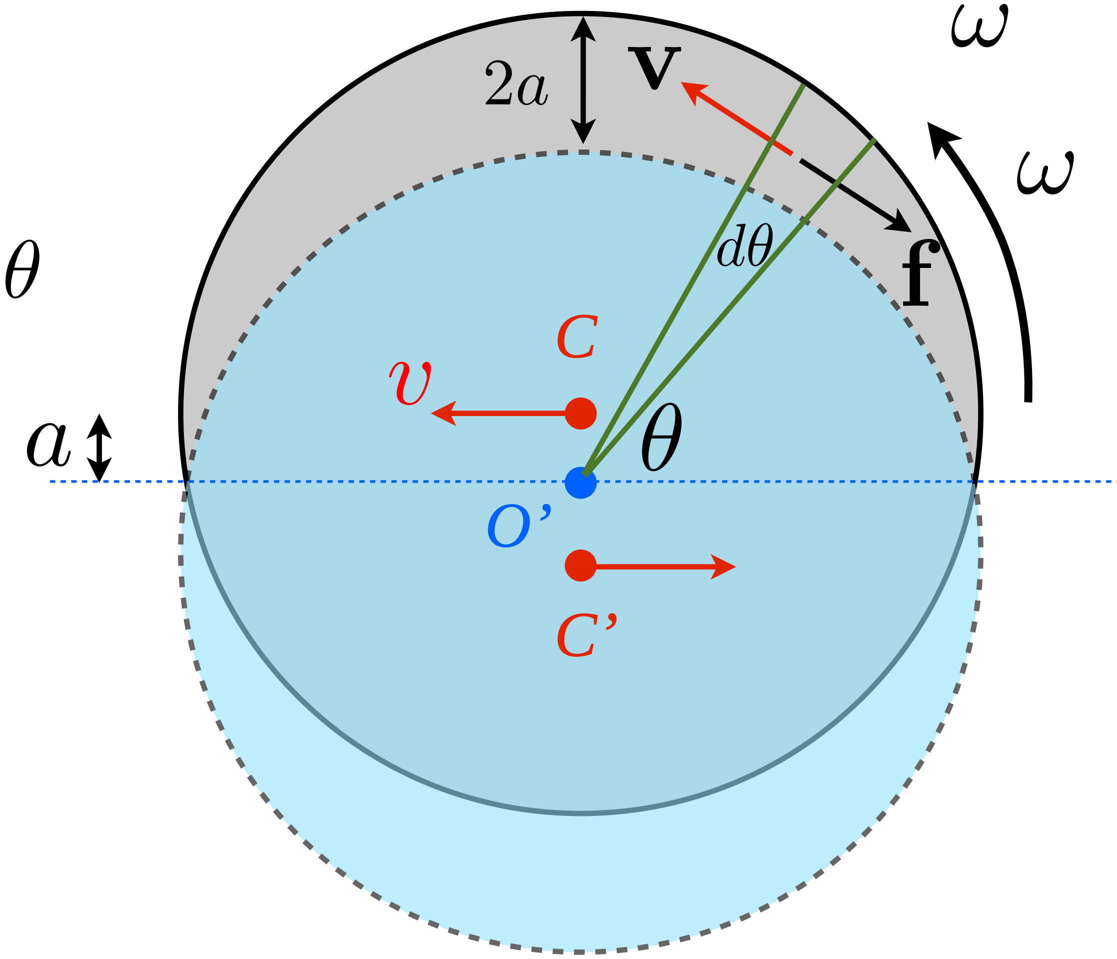
平面运动和瞬心

$$v \ll r\omega \qquad a = \frac{v}{\omega}$$



面密度： $\sigma = \dfrac{m}{\pi r^2}$

月牙面积： $dS = r d\theta \cdot 2a \sin\theta = 2ar \sin\theta d\theta$

*dS*上的摩擦力： $df = \mu\sigma g dS$

摩擦力在速度方向的分量：

$$-df \sin\theta = \frac{2\mu m g a}{\pi r} \sin^2\theta d\theta$$

总摩擦力：

$$f = -\int_0^\pi \sin\theta df = -\frac{2\mu v m g}{\omega\pi r}\int_0^\pi \sin^2\theta d\theta = -\frac{\mu v m g}{\omega r} \propto \frac{v}{\omega}$$

转动惯量

$$\mathbf{L} = \sum_i \mathbf{r}_i \times m_i \mathbf{v}_i = \sum_i \mathbf{r}_i \times m_i (\mathbf{r}_i \times \boldsymbol{\omega})$$

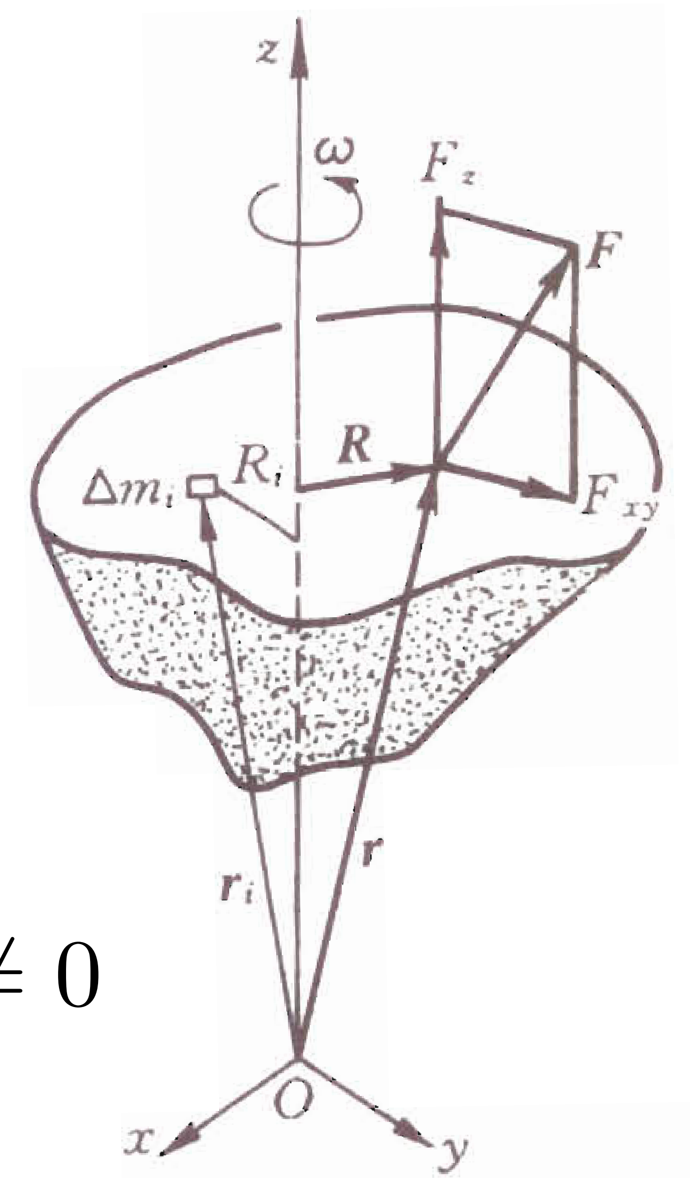
$$\boldsymbol{\omega} = \omega \hat{\mathbf{z}}$$

$$L_z = \mathbf{L} \cdot \hat{\mathbf{z}} = \sum_i m_i \omega \left[\mathbf{r}_i \times (\mathbf{r}_i \times \hat{\mathbf{z}})\right] \cdot \hat{\mathbf{z}}$$

$$= \sum_i m_i \omega \left[(\mathbf{r}_i \cdot \mathbf{r}_i)\hat{\mathbf{z}} - (\mathbf{r}_i \cdot \hat{\mathbf{z}})\mathbf{r}_i\right] \cdot \hat{\mathbf{z}}$$

$$= \sum_i m_i \omega \left[r_i^2 - (\mathbf{r}_i \cdot \hat{\mathbf{z}})^2\right]$$

$$= \omega \sum_i m_i R_i^2 = I_z \omega$$



$$L_{x,y} \neq 0$$

转动惯量： $$I_z = \sum_i m_i R_i^2 = \int R^2 dm = \int_V \rho R^2 dV$$

圆筒的转动惯量

$$I = \int r^2\, dm = \int_{R_1}^{R_2} r^2 \rho(2\pi r L\, dr)$$

$$= 2\pi\rho L \int_{R_1}^{R_2} r^3\, dr$$

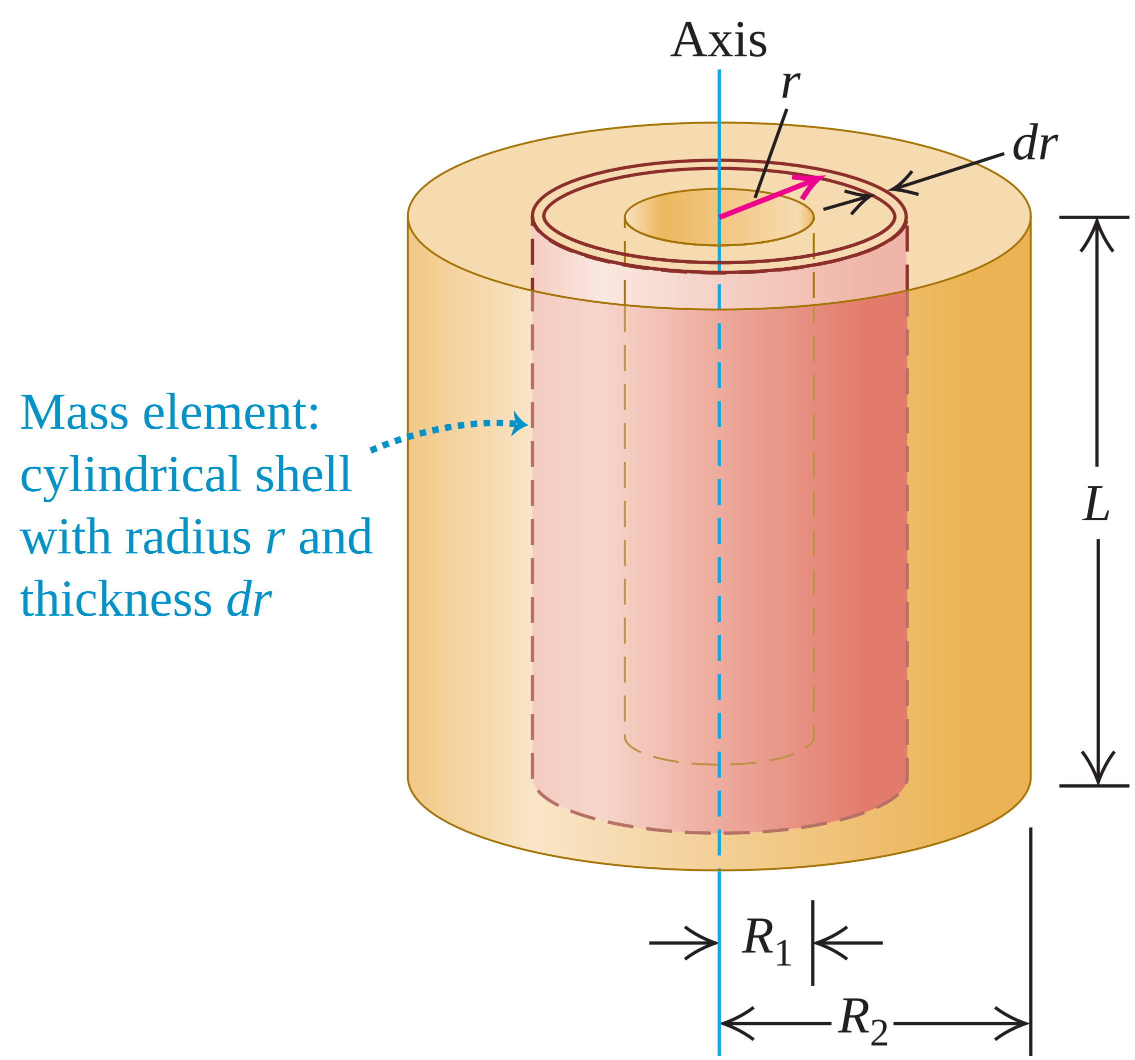
$$= \frac{2\pi\rho L}{4}(R_2^4 - R_1^4)$$

$$= \frac{\pi\rho L}{2}(R_2^2 - R_1^2)(R_2^2 + R_1^2)$$

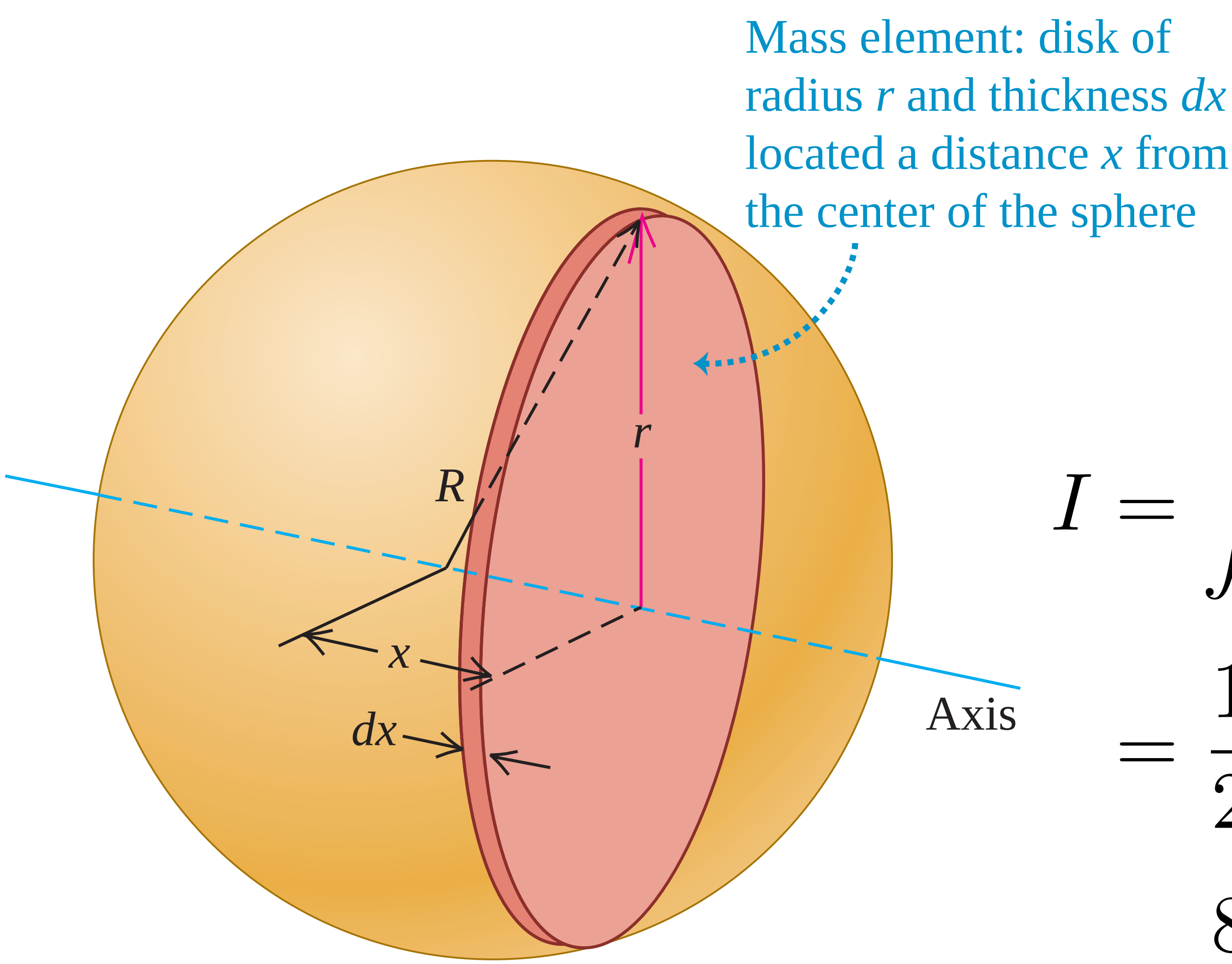
$$I = \tfrac{1}{2}M(R_1^2 + R_2^2)$$

$$dm = \rho\, dV = \rho(2\pi r L\, dr)$$

$$M = \rho V = \pi L \rho(R_2^2 - R_1^2)$$



圆球的转动惯量



$$dI = \frac{1}{2}dm(R_1^2 + R_2^2)$$

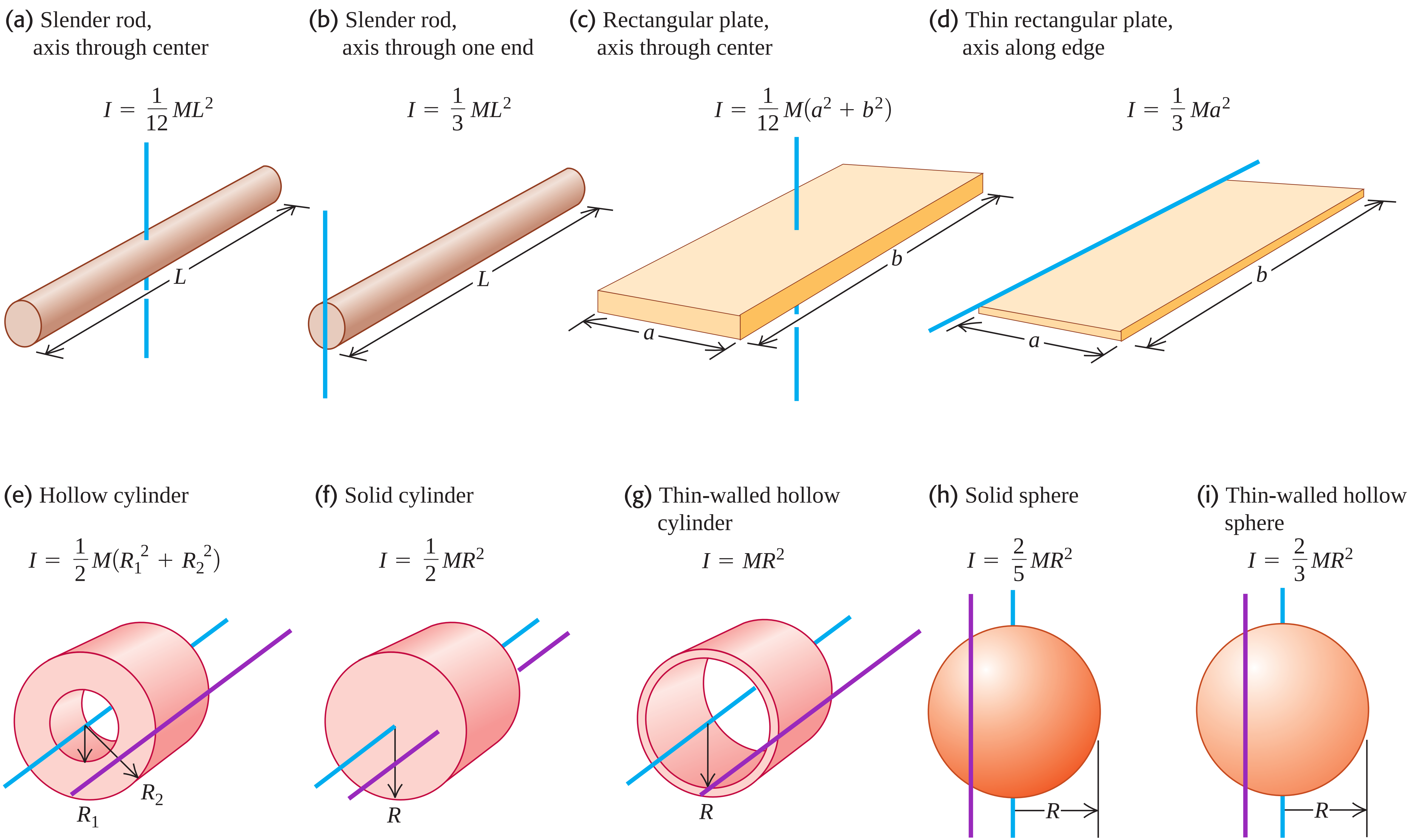
$$= \frac{1}{2}(\rho\pi r^2 dx)r^2 = \frac{1}{2}\pi\rho r^4 dx$$

$$I = \int dI = \frac{1}{2}\pi\rho\int_{-R}^{R} r^4 dx$$

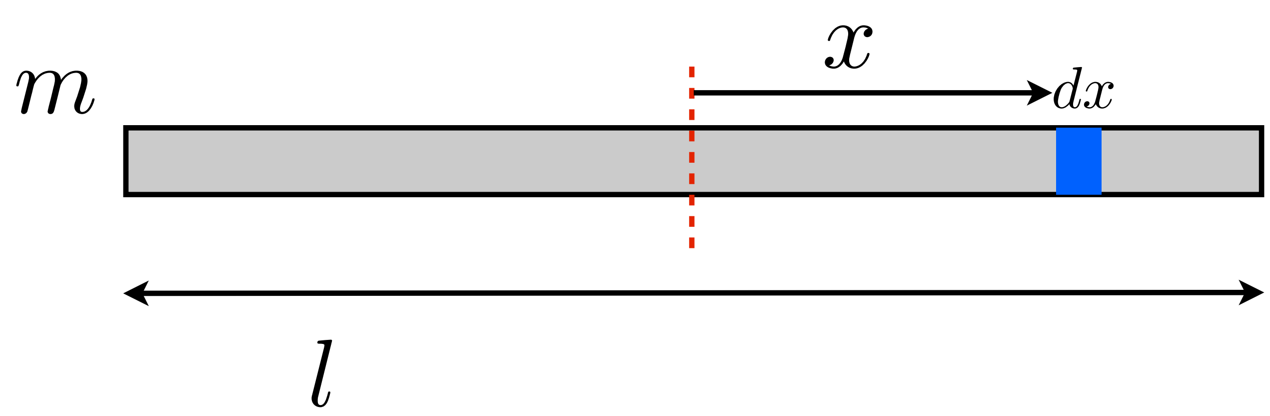
$$= \frac{1}{2}\pi\rho\int_{-R}^{R}(R^2 - x^2)^2 dx = \frac{8\pi\rho R^5}{15}$$

$$= \frac{8\pi R^5}{15}\frac{M}{4\pi R^3/3} = \frac{2}{5}MR^2$$

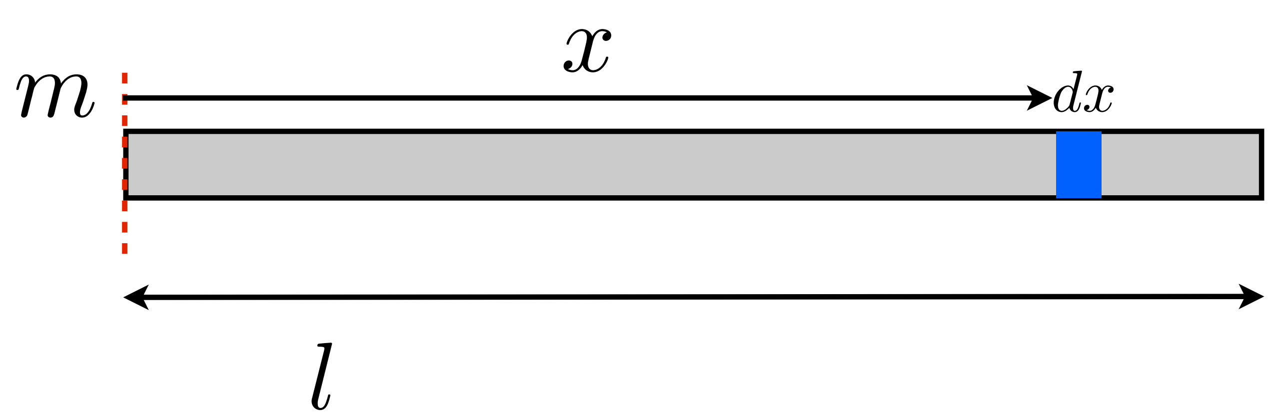
转动惯量


(a) Slender rod, axis through center
$I = \frac{1}{12}ML^2$
L
(b) Slender rod, axis through one end
$I = \frac{1}{3}ML^2$
L
(c) Rectangular plate, axis through center
$I = \frac{1}{12}M(a^2 + b^2)$
a
b
(d) Thin rectangular plate, axis along edge
$I = \frac{1}{3}Ma^2$
a
b
(e) Hollow cylinder
$I = \frac{1}{2}M(R_1^2 + R_2^2)$
R_1
R_2
(f) Solid cylinder
$I = \frac{1}{2}MR^2$
R
(g) Thin-walled hollow cylinder
$I = MR^2$
R
(h) Solid sphere
$I = \frac{2}{5}MR^2$
R
(i) Thin-walled hollow sphere
$I = \frac{2}{3}MR^2$
R

转动惯量

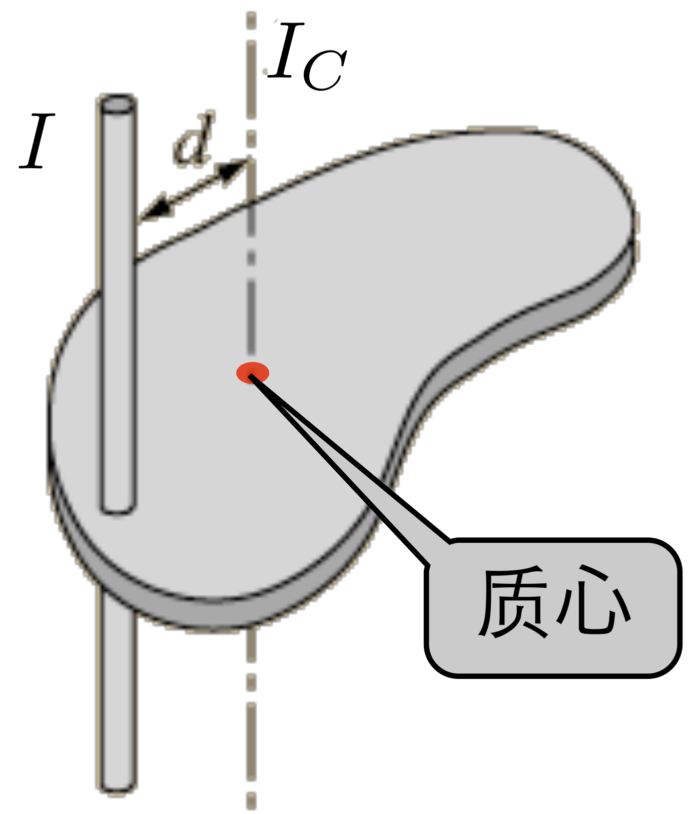


$$I = \int x^2 dm = \int_{-l/2}^{l/2} x^2 \frac{m}{l} dx = \frac{m}{l}\frac{1}{3}x^3\Big|_{-l/2}^{l/2} = \frac{ml^2}{12}$$



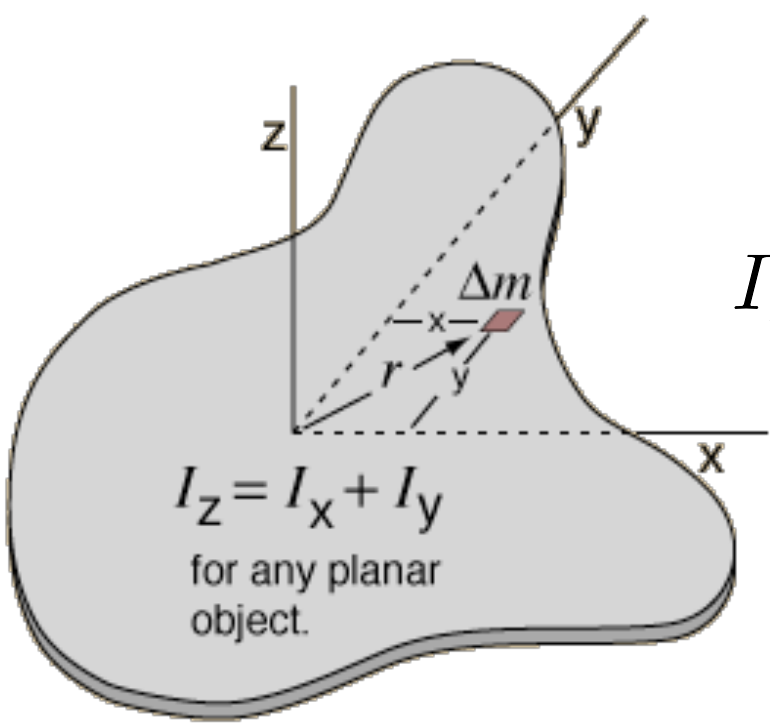
$$I = \int x^2 dm = \int_0^l x^2 \frac{m}{l} dx = \frac{m}{l}\frac{1}{3}x^3\Big|_0^l = \frac{ml^2}{3} = \frac{ml^2}{12} + m\left(\frac{l}{2}\right)^2$$

平行轴定理和薄板正交轴定理



$$I_C = \int (x^2 + y^2)\, dm$$

$$\begin{aligned} I &= \int \left[x^2 + (y+d)^2\right] dm \\ &= I_C + \int (2dy) dm + \int d^2 dm \\ &= I_C + M d^2 \end{aligned}$$



$$I_z = \int (x^2 + y^2) dm = \int x^2 dm + \int y^2 dm = I_x + I_y$$

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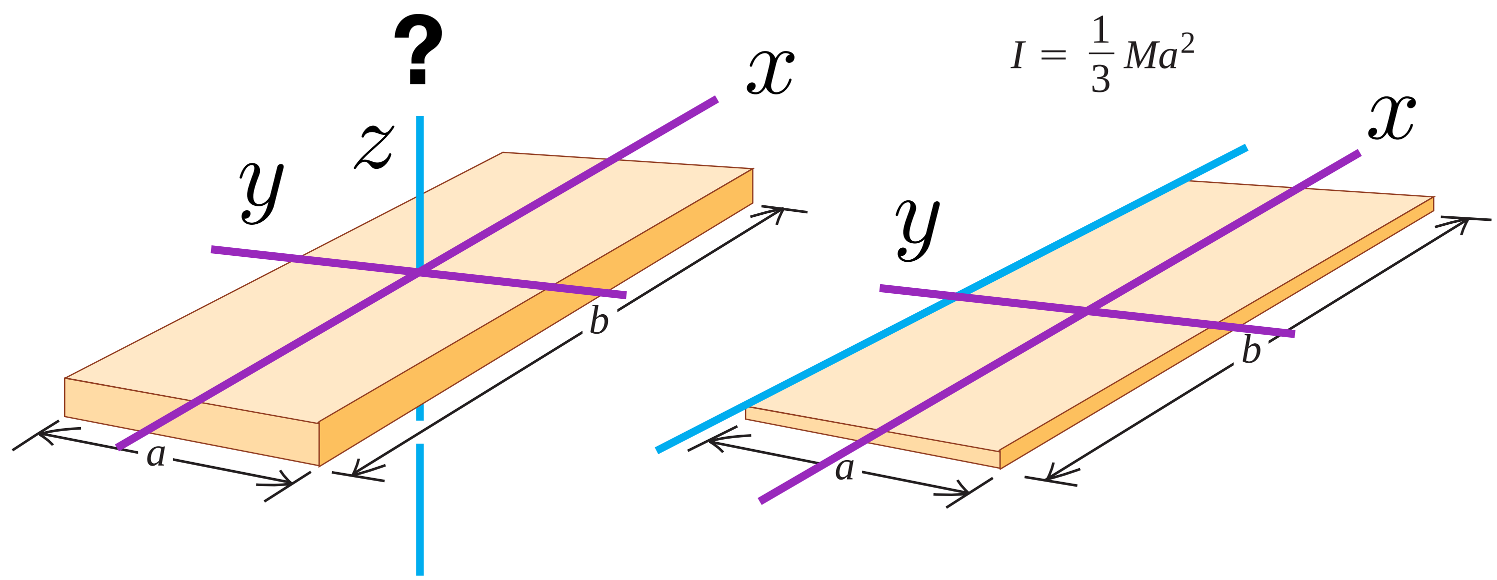
平行轴定理和薄板正交轴定理



平行轴定理： $I = I_x + M\left(\frac{a}{2}\right)^2 \quad \Rightarrow \quad I_x = \frac{1}{12}Ma^2$

$$I_y = \frac{1}{12}Mb^2$$

薄板正交轴定理：$I_z = I_x + I_y = \frac{1}{12}M\left(a^2 + b^2\right)$

立方体的转动惯量

定轴转动定理

$$\frac{d\mathbf{L}}{dt} = \mathbf{r} \times \mathbf{F} = \mathbf{T}$$

$$\frac{dL_z}{dt} = \frac{d(I\omega)}{dt} = I\frac{d\omega}{dt} = T_z$$

$$T_z = (\mathbf{r} \times \mathbf{F})_z = xF_y - yF_x$$

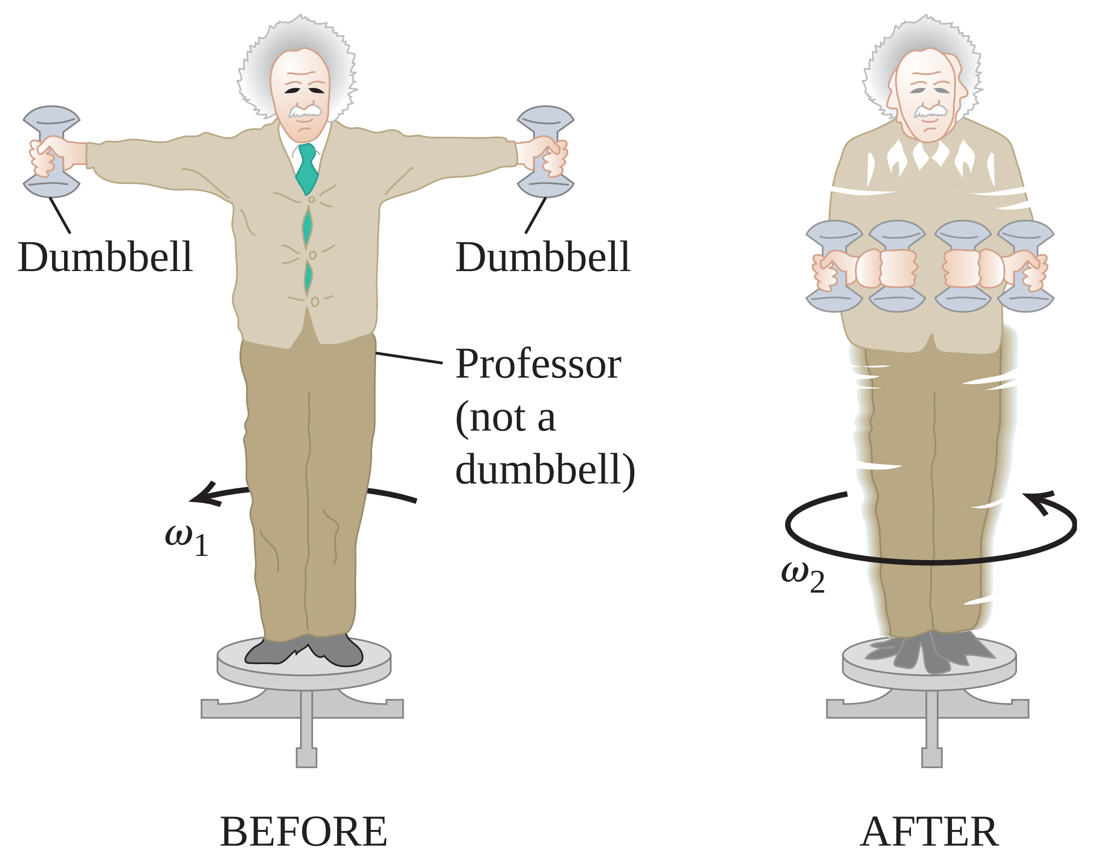
角动量守恒

$$\frac{d\mathbf{L}}{dt} = \mathbf{r} \times \mathbf{F} = \mathbf{T}$$

$$\frac{dL_z}{dt} = \frac{d(I\omega)}{dt} = I\frac{d\omega}{dt} = T_z$$

$$T_z = (\mathbf{r} \times \mathbf{F})_z = xF_y - yF_x$$

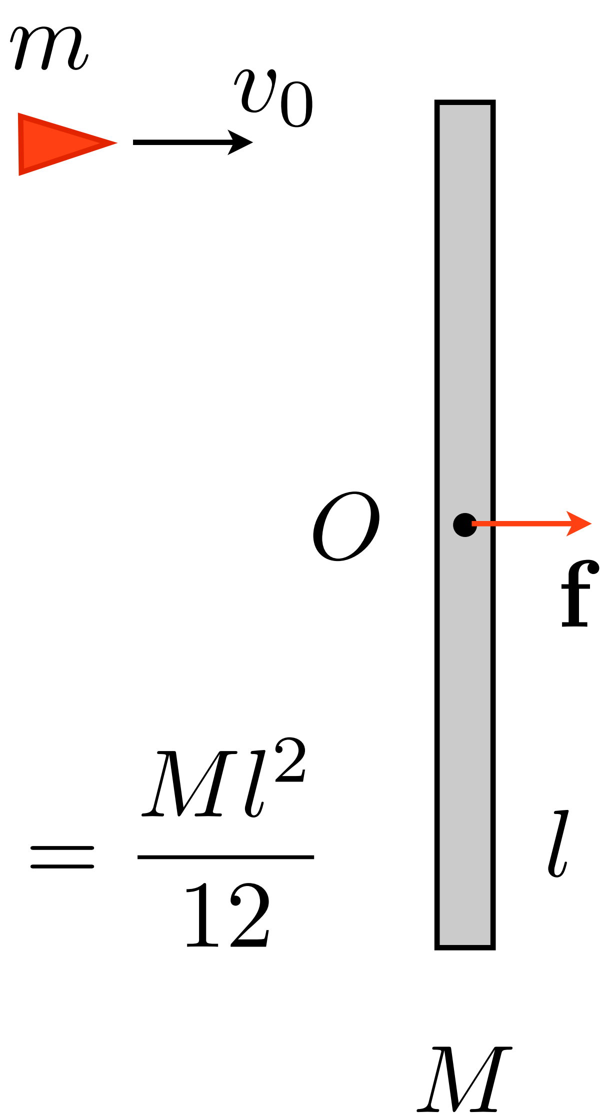
$$T_z = 0 \quad \Rightarrow \quad I\omega = \text{const.}$$



跳水：http://v.youku.com/v_show/id_XMjg4MDI5MDY0.html

花样滑冰：http://v.youku.com/v_show/id_XMjYxMDQ3Mzk2.html

角动量守恒



$$I = \frac{Ml^2}{12}$$

$$mv_0 \frac{l}{2} = \left[I + m \left(\frac{l}{2} \right)^2 \right] \omega$$

$$\omega = \frac{6mv_0}{(M+3m)l}$$

$$f \Delta t = \Delta p = M \cdot 0 + m \frac{l}{2} \omega - mv_0$$
$$= \left(\frac{3m}{M+3m} - 1 \right) mv_0 = \frac{M}{M+3m} mv_0$$

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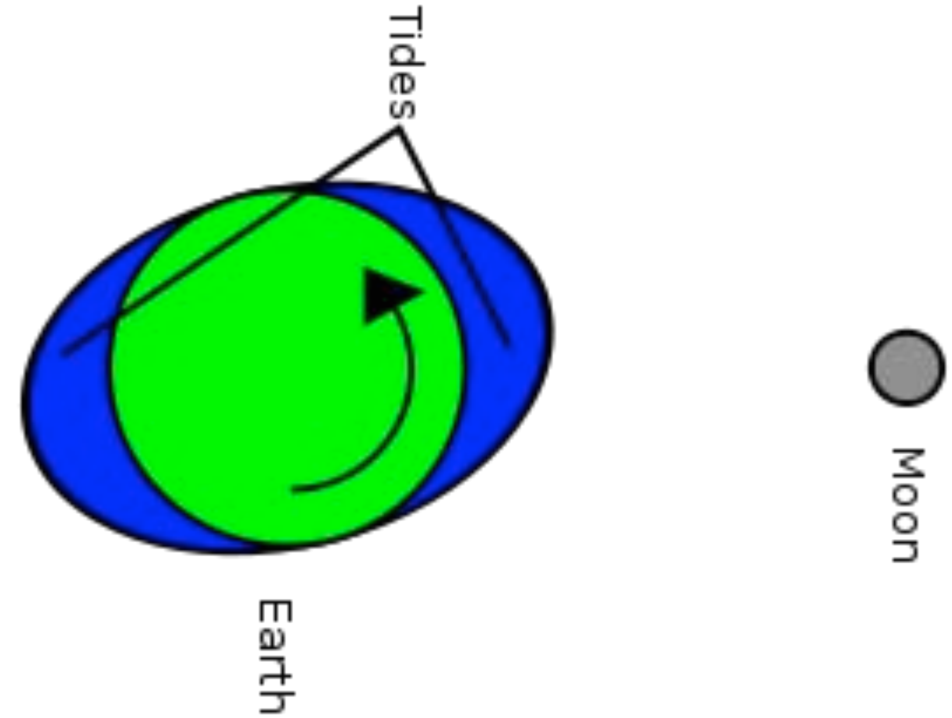
明天太阳会照常升起吗？

今天日出：05:46

今天日落：18:11



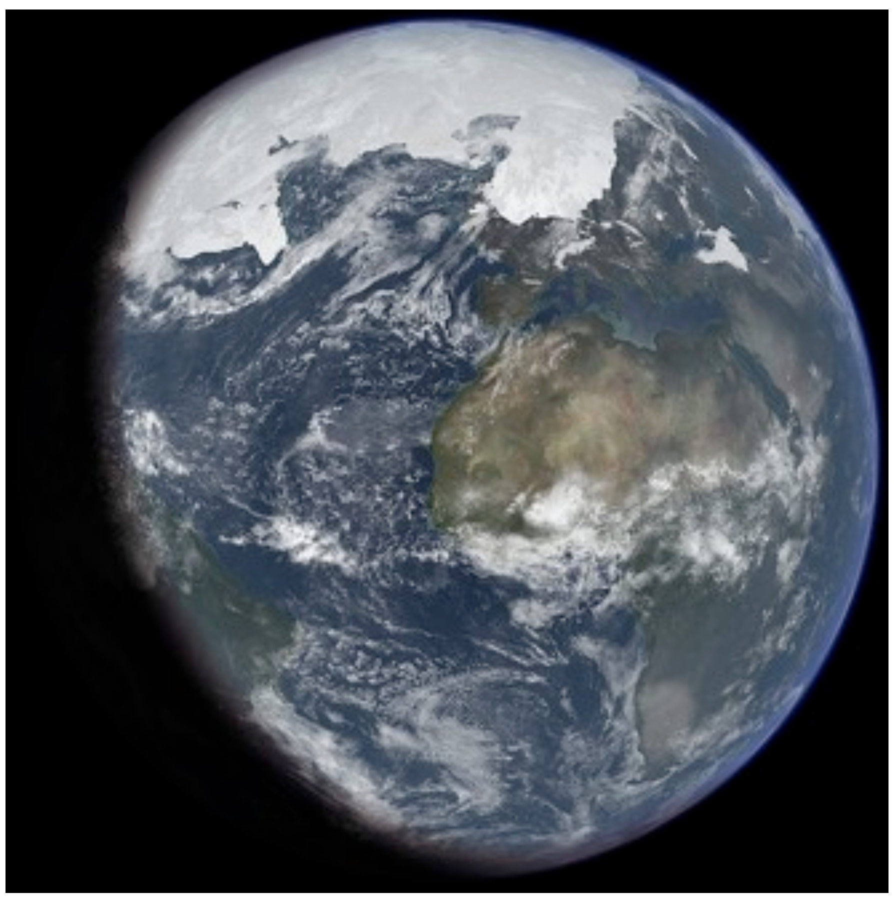
$$\sum \mathbf{L}_i = \text{const.}$$

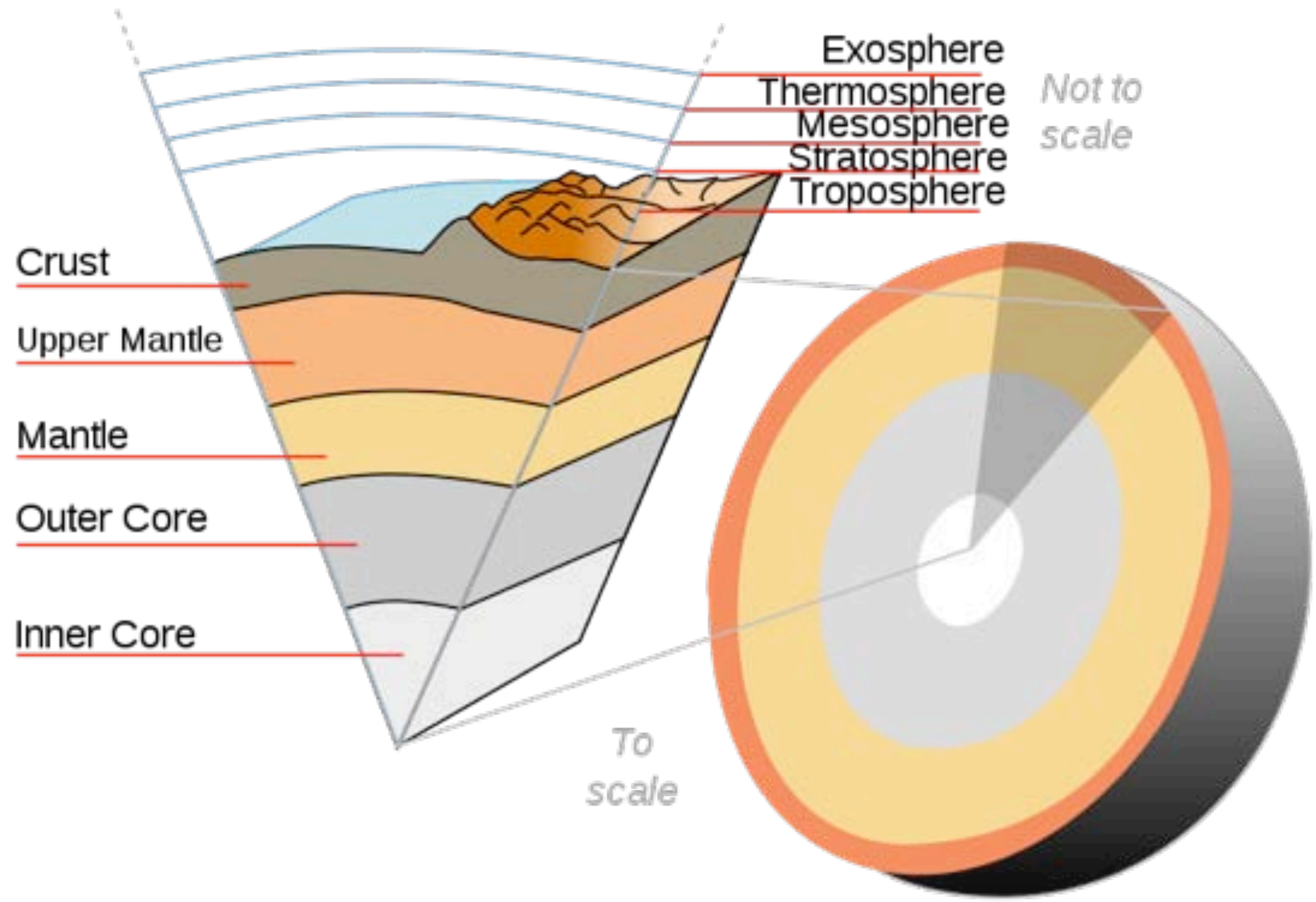
月亮远离地球：4厘米/年

地球每天变长：23毫秒

角动量守恒与冰河时期、地幔

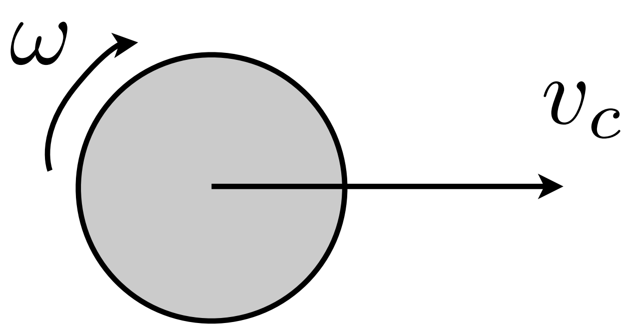
$$L = I\omega = \text{const.}$$





定轴转动的动能定理

直线运动+转动



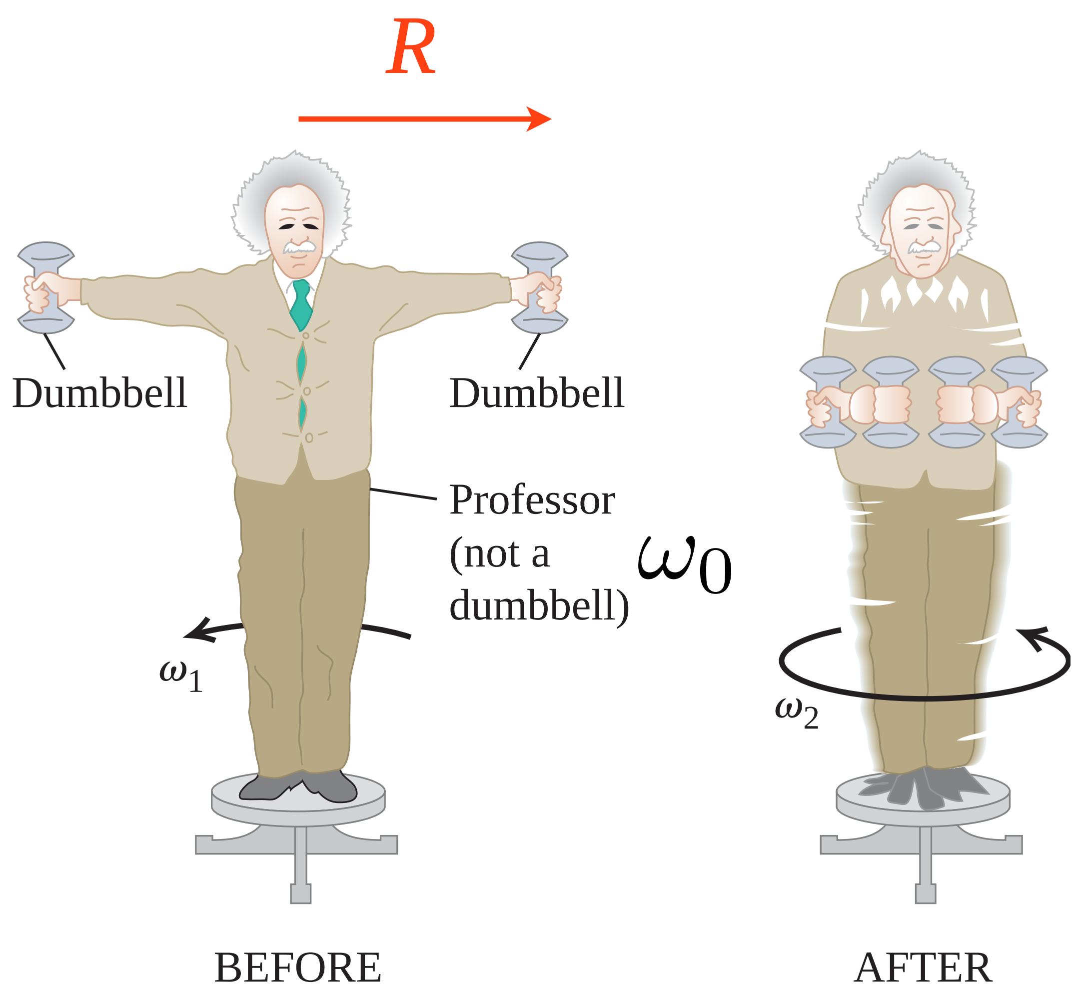
$$E_k = \frac{1}{2}mv_c^2 + \frac{1}{2}I_c\omega^2$$

$$\begin{aligned}E_k &= \sum_i \frac{1}{2}m_i v_i^2 = \sum_i \frac{1}{2}m_i(\omega r_i)^2 \\ &= \sum_i \frac{1}{2}m_i r_i^2\omega^2 = \frac{1}{2}\left(\sum_i m_i r_i^2\right)\omega^2 \\ &= \frac{1}{2}I\omega^2\end{aligned}$$

飞轮储能

$$W = \int T d\phi = \frac{1}{2}I\omega^2 - \frac{1}{2}I\omega_0^2$$

定轴转动的动能定理



ω_R ω_0

$$I_0\omega_0 = I_R\omega_R$$

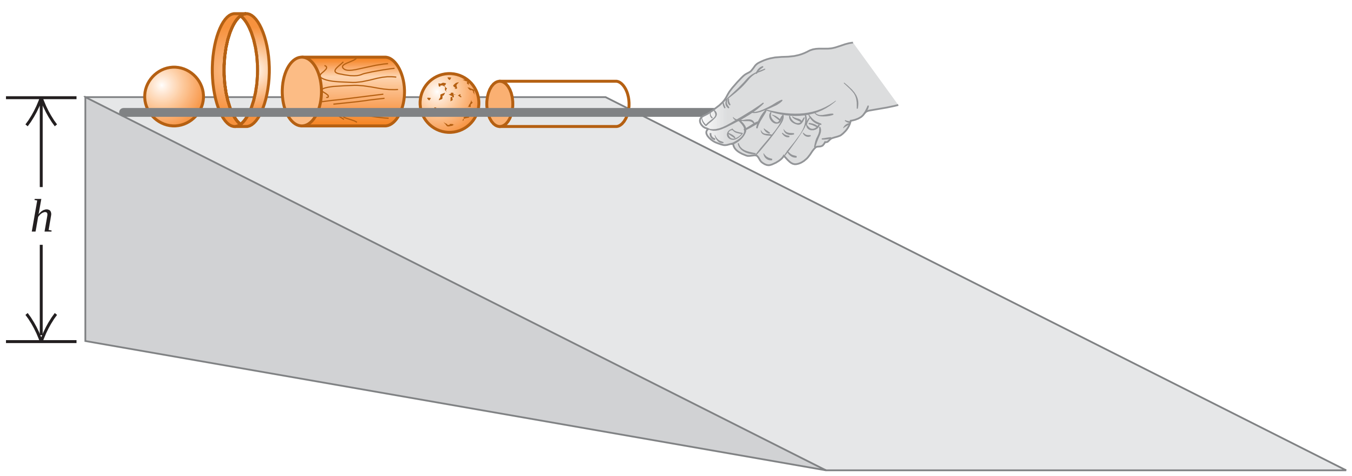
$$\frac{1}{2}I_0\omega_0^2 \overset{?}{\lessgtr} \frac{1}{2}I_R\omega_R^2$$

$$I_R = I_0 + mR^2 \qquad I_0$$

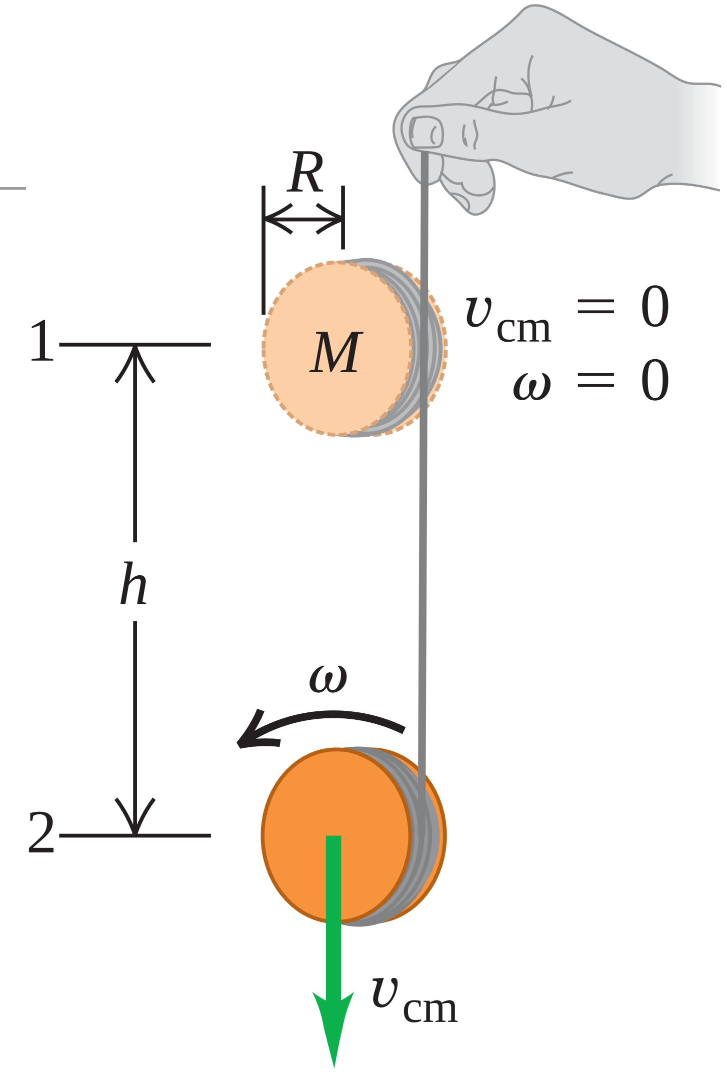
$$W = \frac{1}{2}I_0\omega_0^2 - \frac{1}{2}I_R\omega_R^2 < 0$$

一些问题

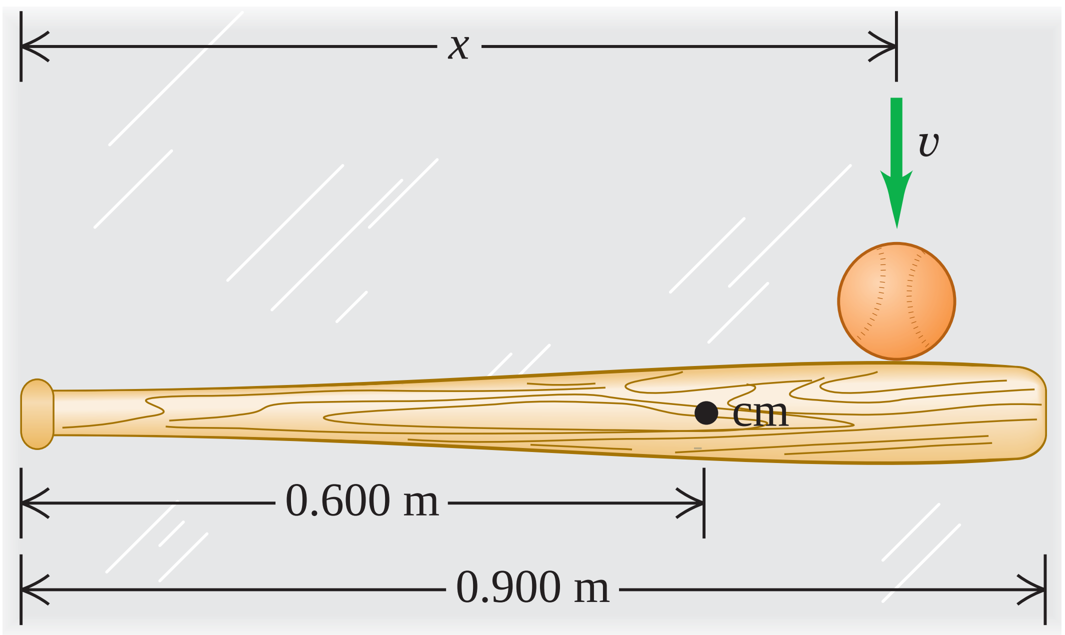
哪个滚的最快



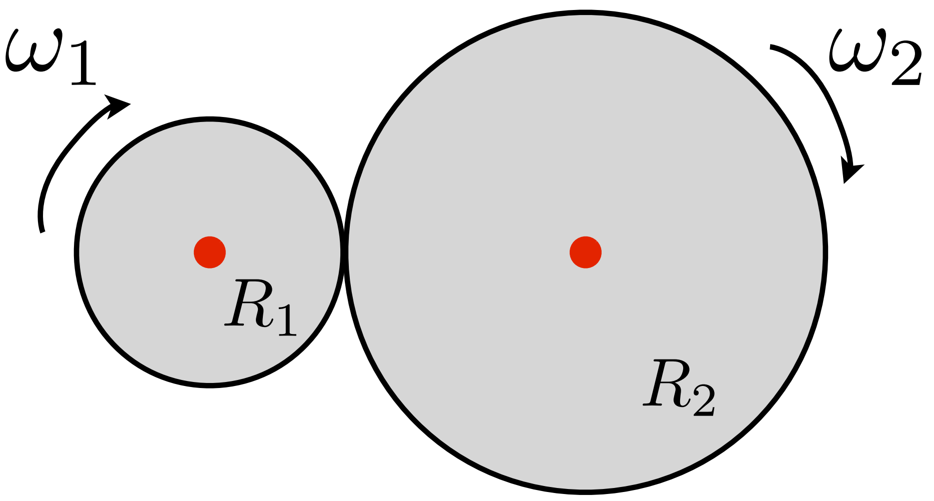
悠悠球



击球中心



两轮磨合

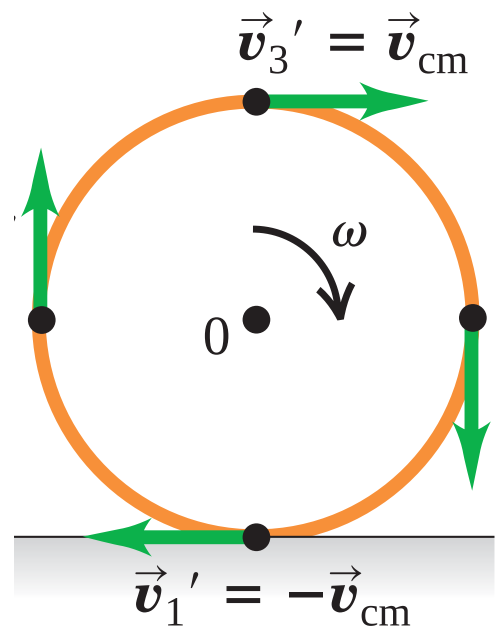


一些问题

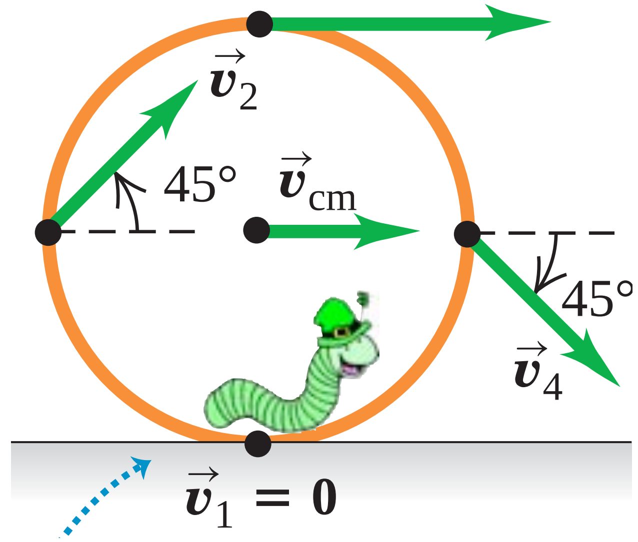
轮子滚动问题

纯滚到滚动

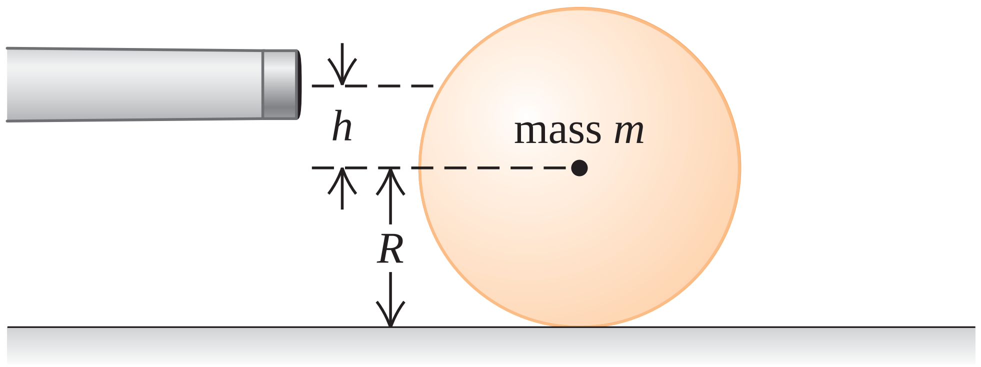
纯滑到滚动



人行走问题



台球



汽车前进问题

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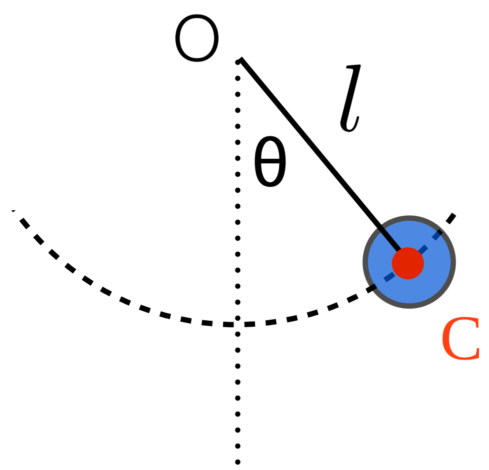
一些问题

$$g_{\text{eff}} = g$$

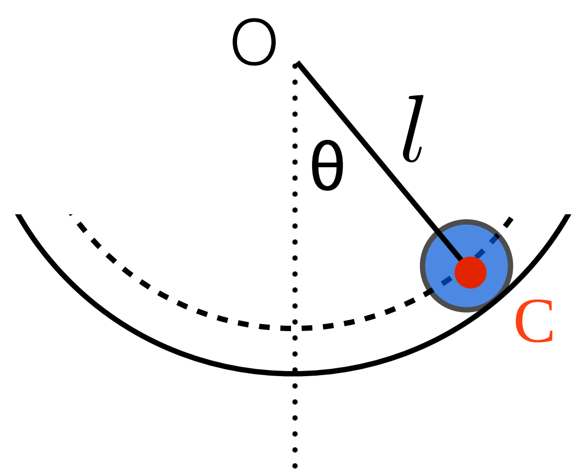
单摆



$$mgl(1-\cos\theta) = \frac{1}{2}mv_c^2$$

$$g_{\text{eff}} = \frac{g}{1+c}$$

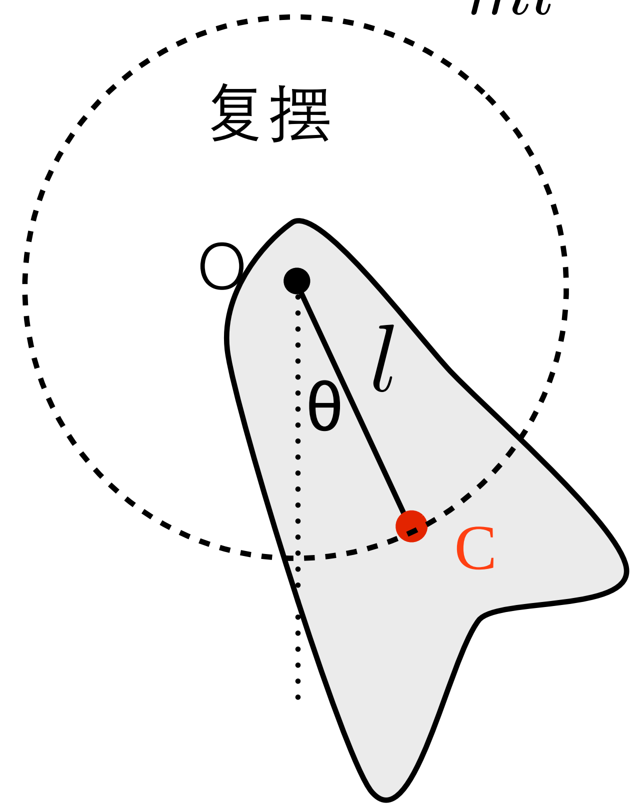
球形碗中的小球周期



$$\begin{aligned} &mgl(1-\cos\theta) \\ =&\frac{1}{2}mv_c^2 + \frac{1}{2}I\omega^2 \\ =&\frac{1}{2}m(1+c)v_c^2 \end{aligned}$$

$$g_{\text{eff}} = \frac{g}{1+\frac{I_C}{ml^2}}$$

复摆



$$\begin{aligned} &mgl(1-\cos\theta) \\ =&\frac{1}{2}I\omega^2 \\ =&\frac{1}{2}(ml^2+I_C)\omega^2 \\ =&\frac{1}{2}m\left(1+\frac{I_C}{ml^2}\right)v_c^2 \end{aligned}$$

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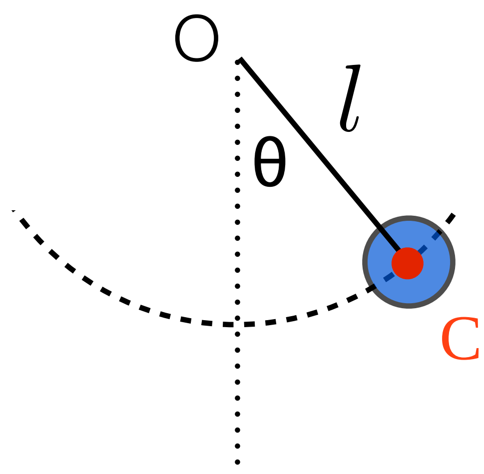
一些问题

$$g_{\text{eff}} = g$$

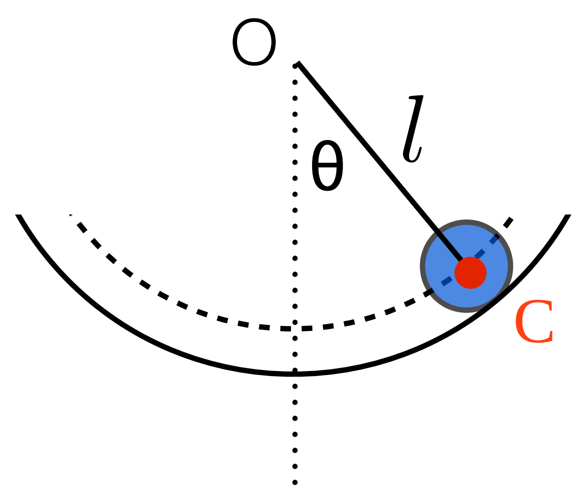
单摆



$$T = 2\pi\sqrt{\frac{l}{g}}$$

$$g_{\text{eff}} = \frac{g}{1+c}$$

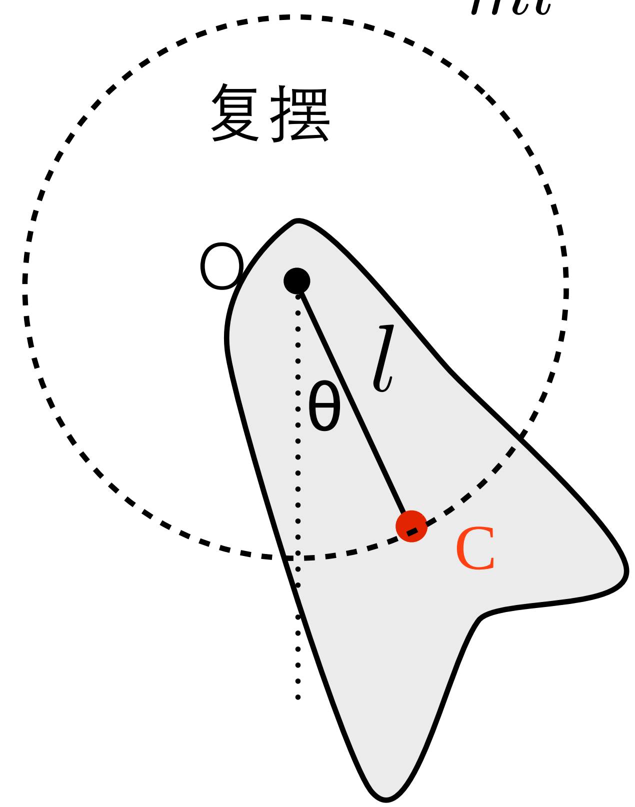
球形碗中的小球周期



$$T = 2\pi\sqrt{\frac{(1+c)l}{g}}$$

$$g_{\text{eff}} = \frac{g}{1+\frac{I_C}{ml^2}}$$

复摆



$$T = 2\pi\sqrt{\frac{l + I_c/ml}{g}}$$

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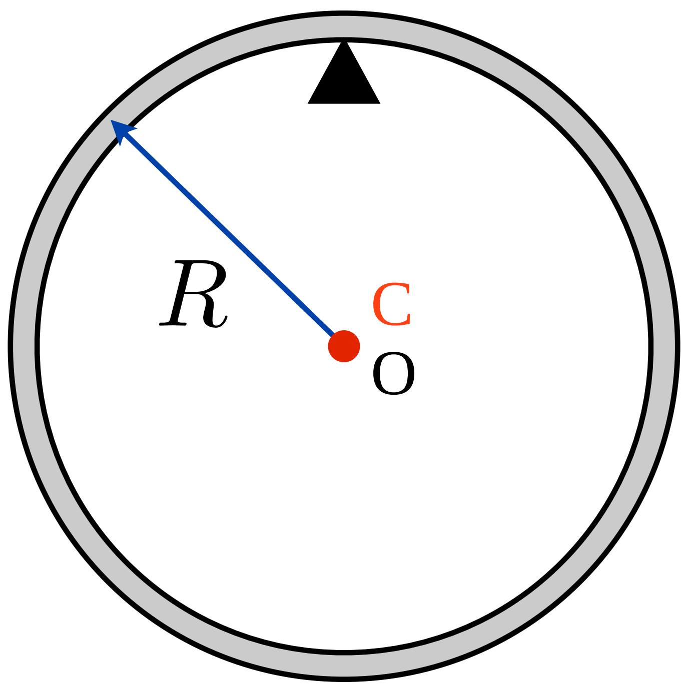
一些问题

$$g_{\text{eff}} = \frac{g}{1 + \frac{I_C}{ml^2}}$$



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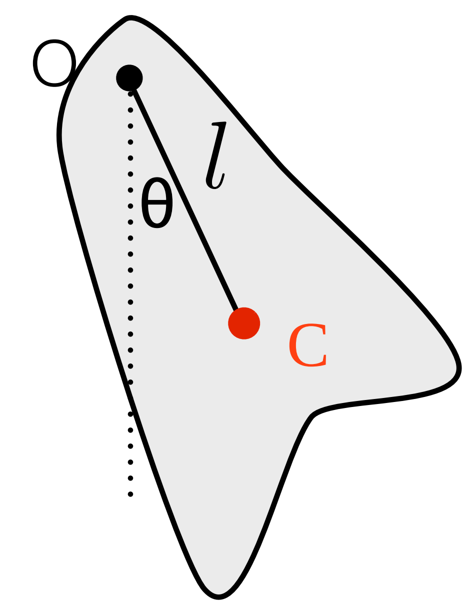
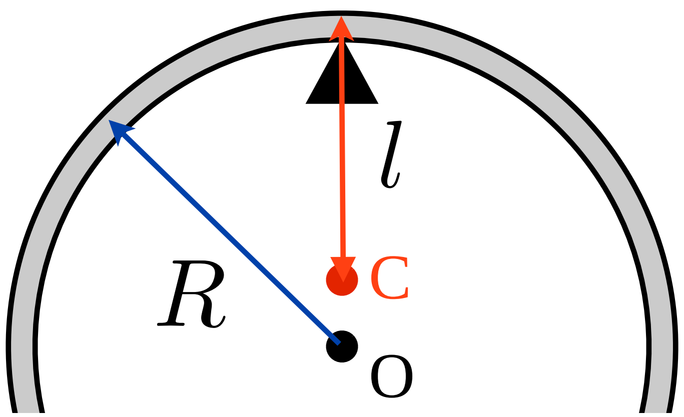


$$T = 2\pi\sqrt{\frac{2R}{g}}$$

$$T = 2\pi\sqrt{\frac{l + I_c/ml}{g}}$$

一些问题



$$I_O = I_C + m(R-l)^2$$

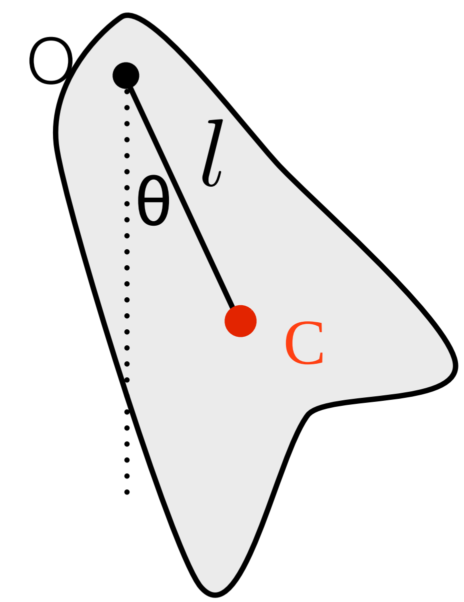
$$I_O = mR^2$$

$$I_C = m(2R-l)l \Rightarrow l + \frac{I_C}{ml} = 2R$$

$$T = 2\pi\sqrt{\frac{2R}{g}}$$

$$g_{\text{eff}} = \frac{g}{1+\frac{I_C}{ml^2}}$$

复摆

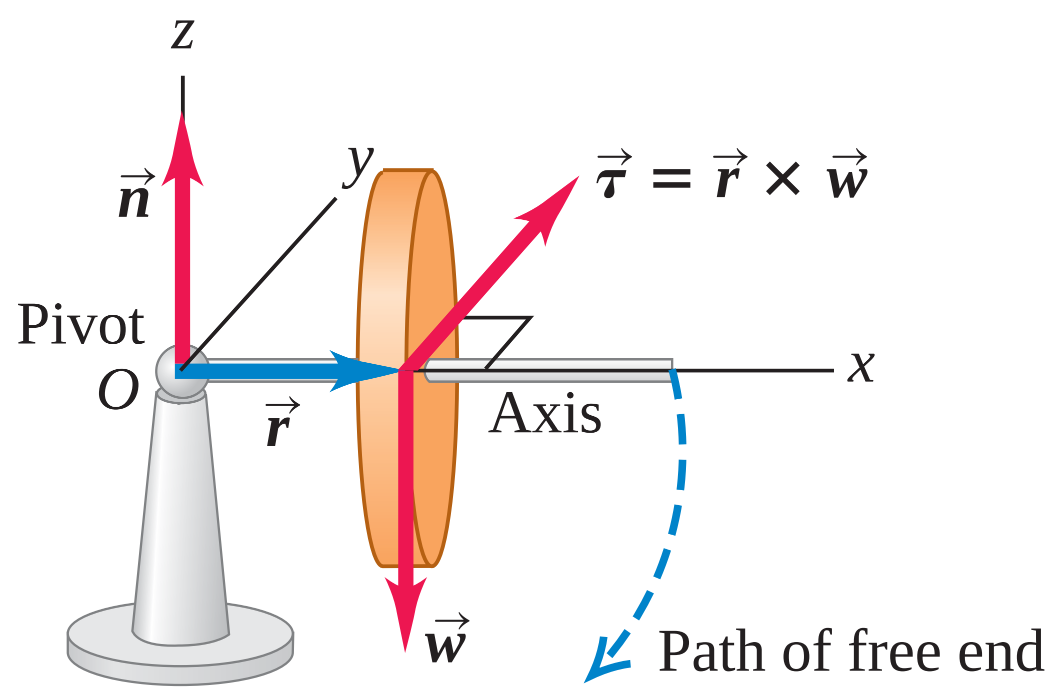


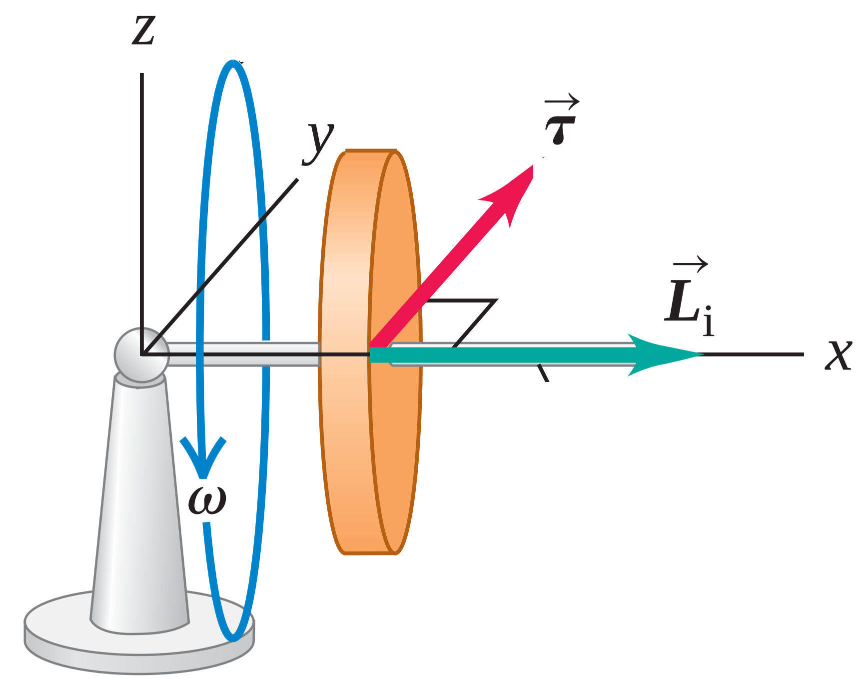
$$T = 2\pi\sqrt{\frac{l + I_c/ml}{g}}$$

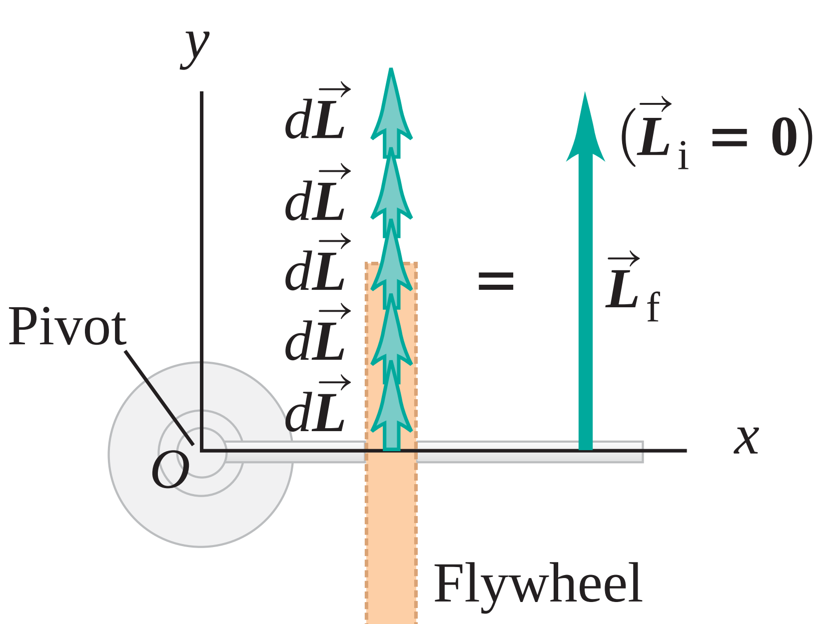
转动物体的稳定性

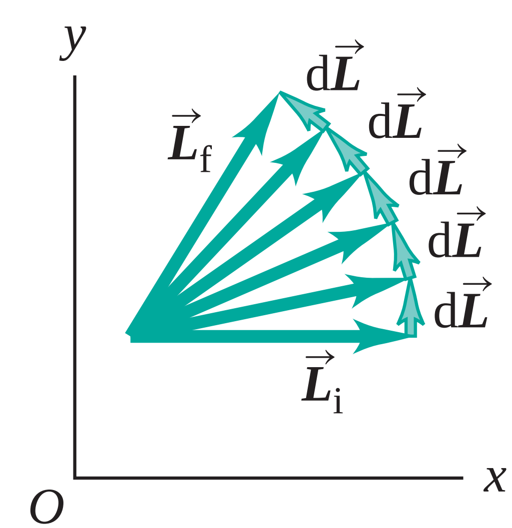
Prepared by Jiang Xiao

陀螺仪







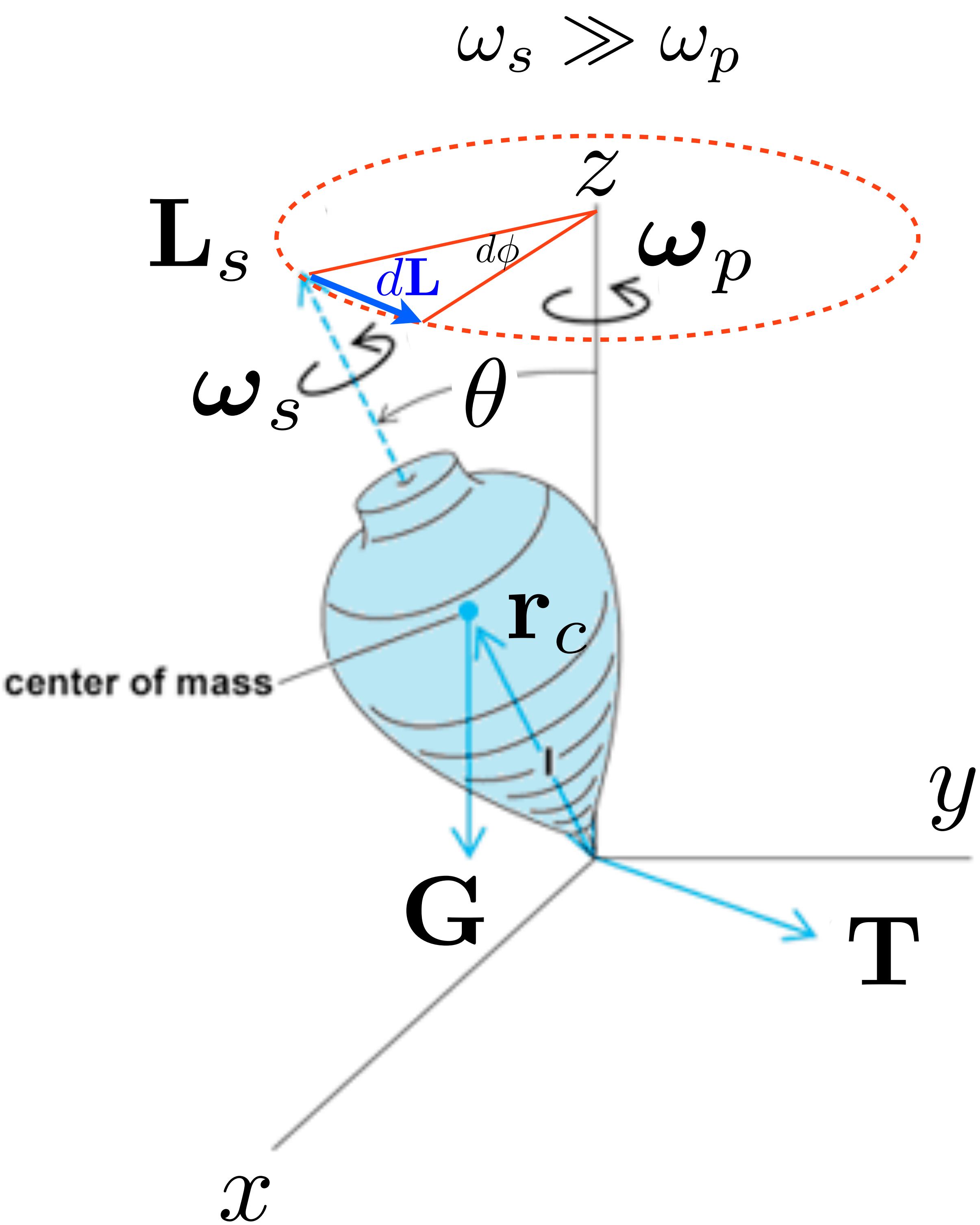


自行车轮子：http://v.youku.com/v_show/id_XMTQxMDc4NDU2.html

陀螺：http://v.youku.com/v_show/id_XMjY5NTgzNTE2.html

自行车转弯，子弹炮弹稳定性

陀螺仪



$$\boldsymbol{\omega} = \boldsymbol{\omega}_s + \boldsymbol{\omega}_p \approx \boldsymbol{\omega}_s$$

$$\mathbf{L} = \mathbf{L}_s + \mathbf{L}_p \approx \mathbf{L}_s = I\boldsymbol{\omega}_s$$

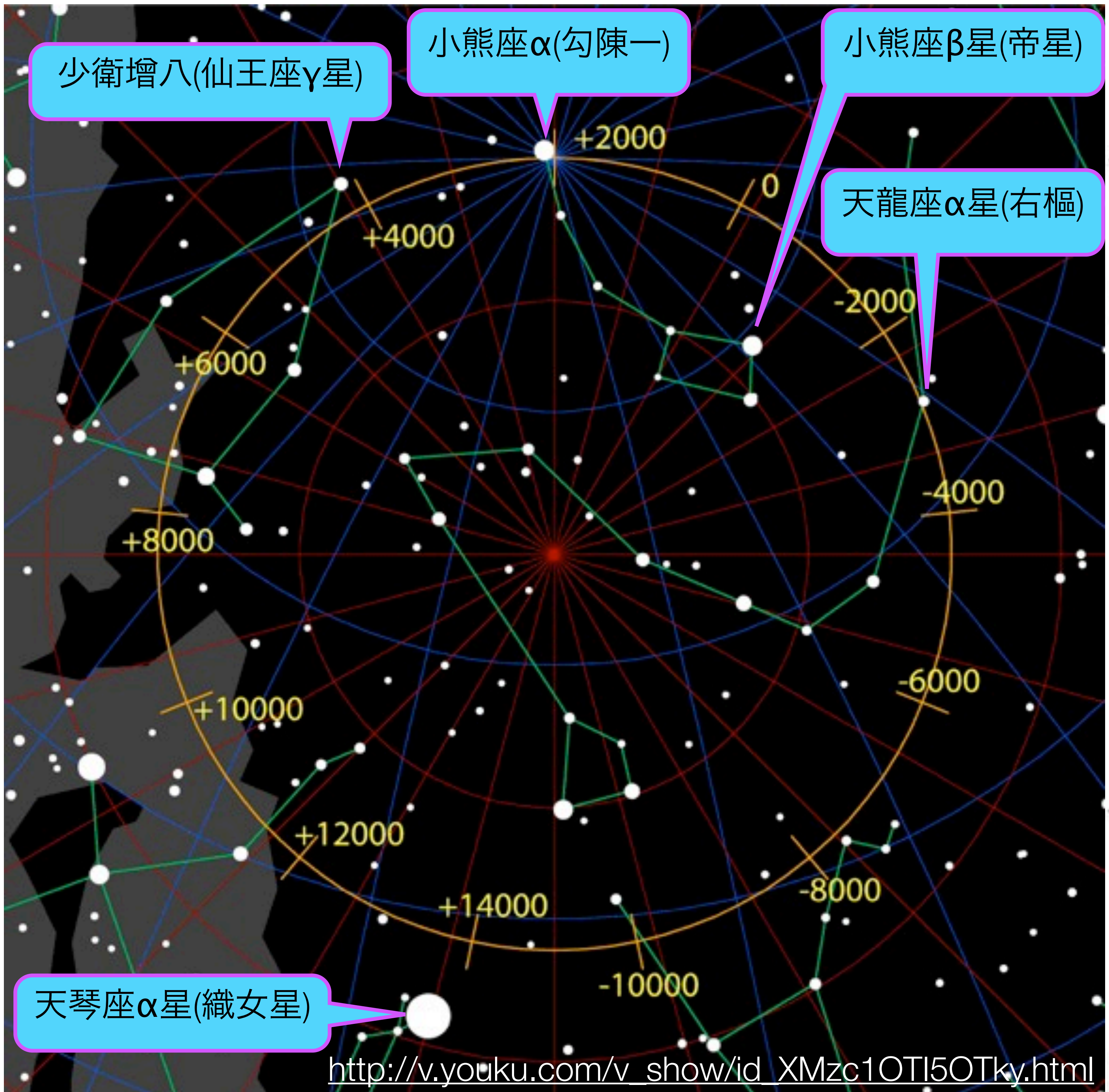
$$\frac{d\mathbf{L}}{dt} = \mathbf{T} = \mathbf{r}_c \times \mathbf{G}$$

$$\frac{dL}{dt} = L_s \sin\theta \frac{d\phi}{dt} = I\omega_s\omega_p \sin\theta$$

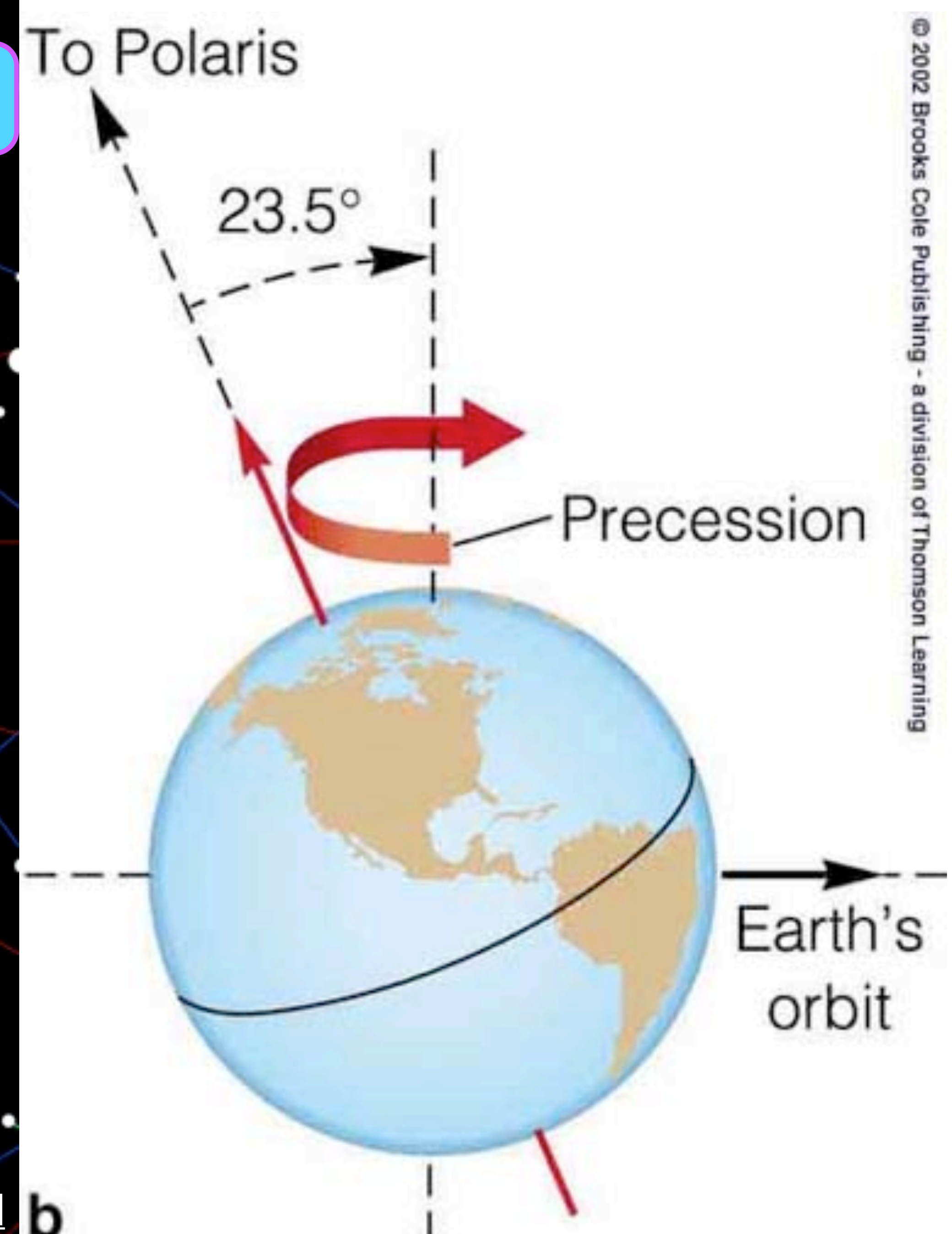
$$T = mgr_c \sin\theta$$

$$\omega_p \approx \frac{mgr_c}{I\omega_s}$$

地球的进动


少衛增八(仙王座γ星)
小熊座α(勾陳一)
小熊座β星(帝星)
天龍座α星(右樞)
天琴座α星(織女星)
+2000
0
+4000
-2000
+6000
-4000
+8000
+10000
-6000
+12000
-8000
+14000
-10000
http://v.youku.com/v_show/id_XMzc1OTI5OTky.html

岁差、北极星


To Polaris
23.5°
Precession
Earth's orbit
b
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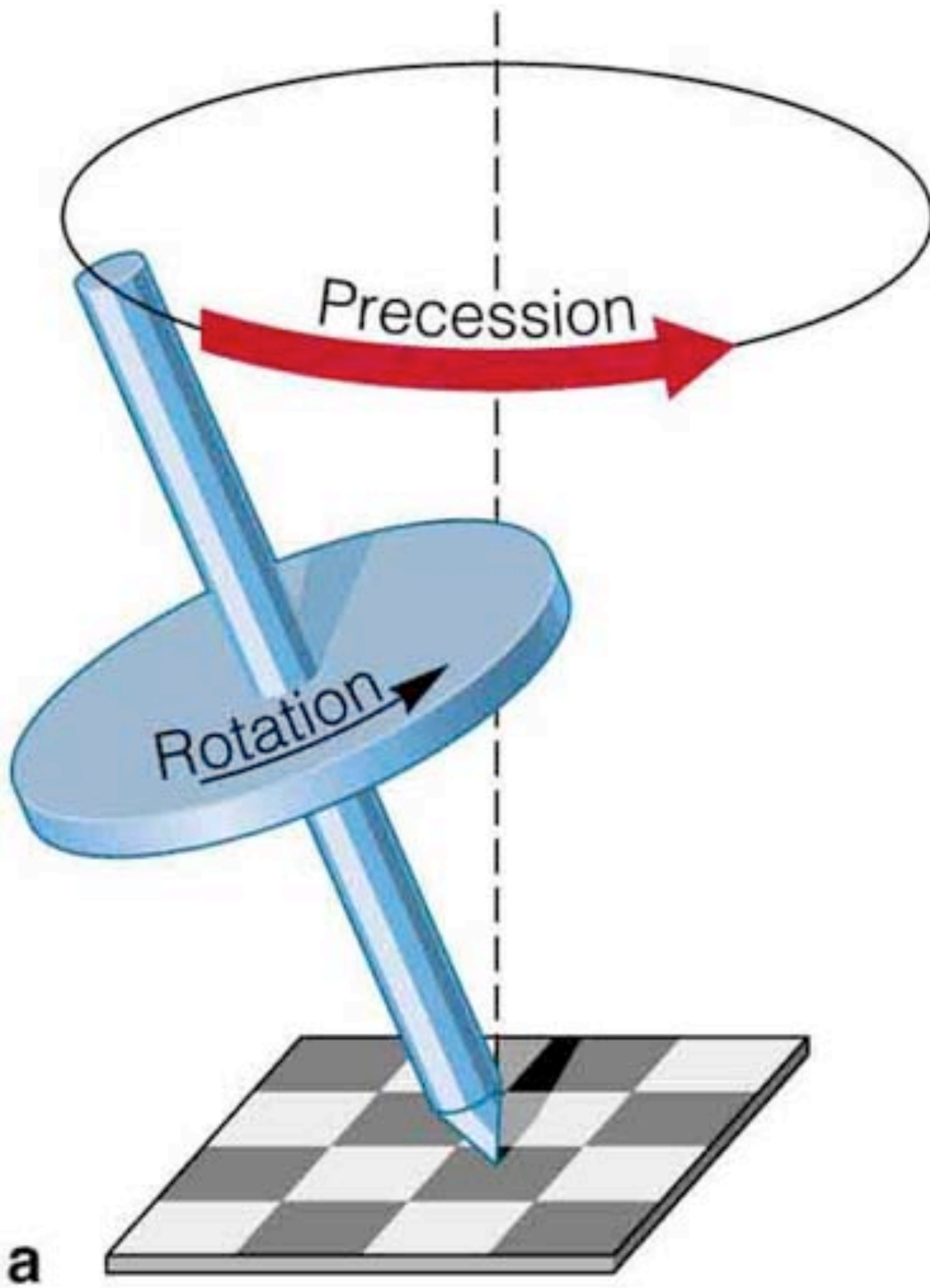
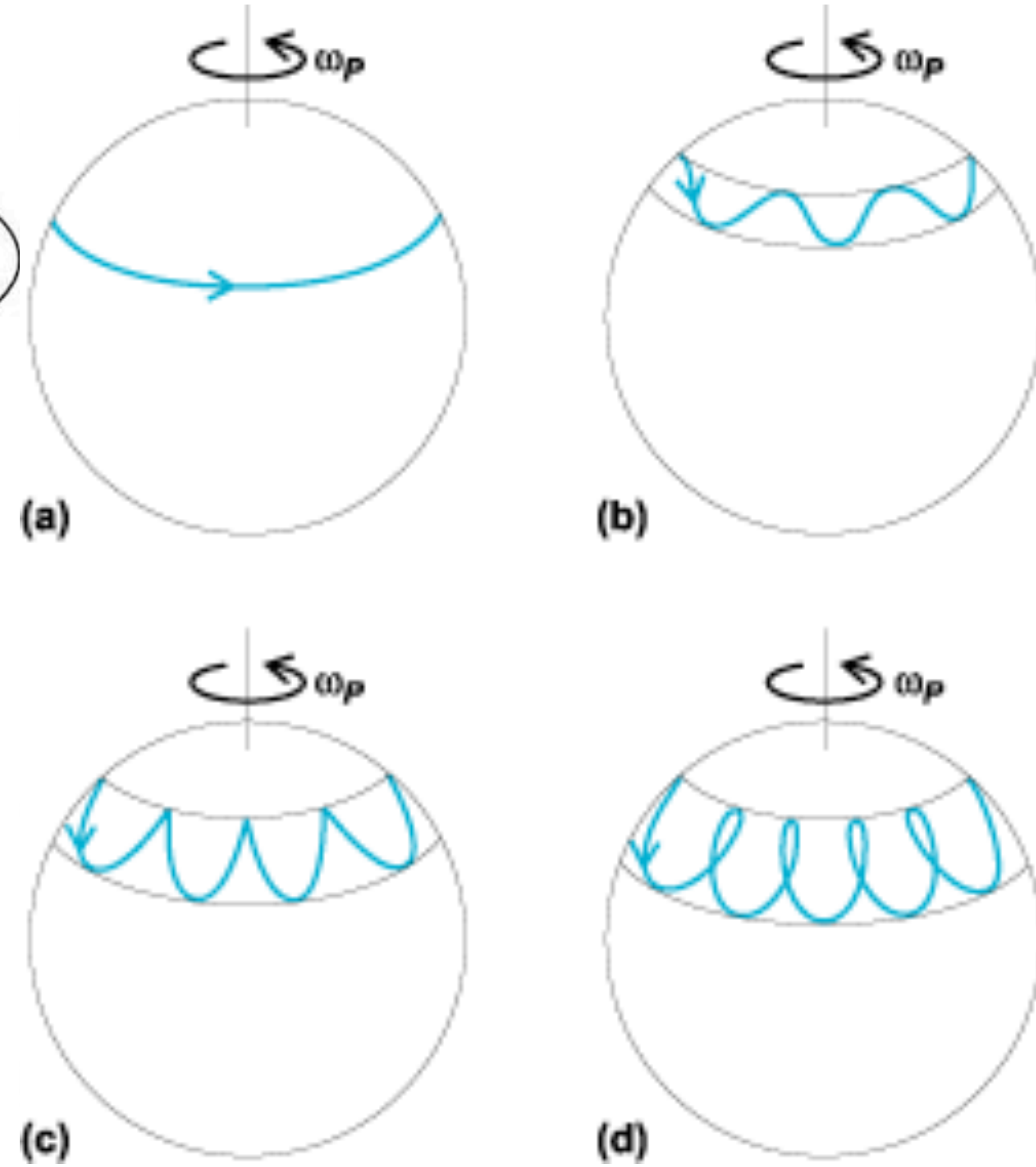
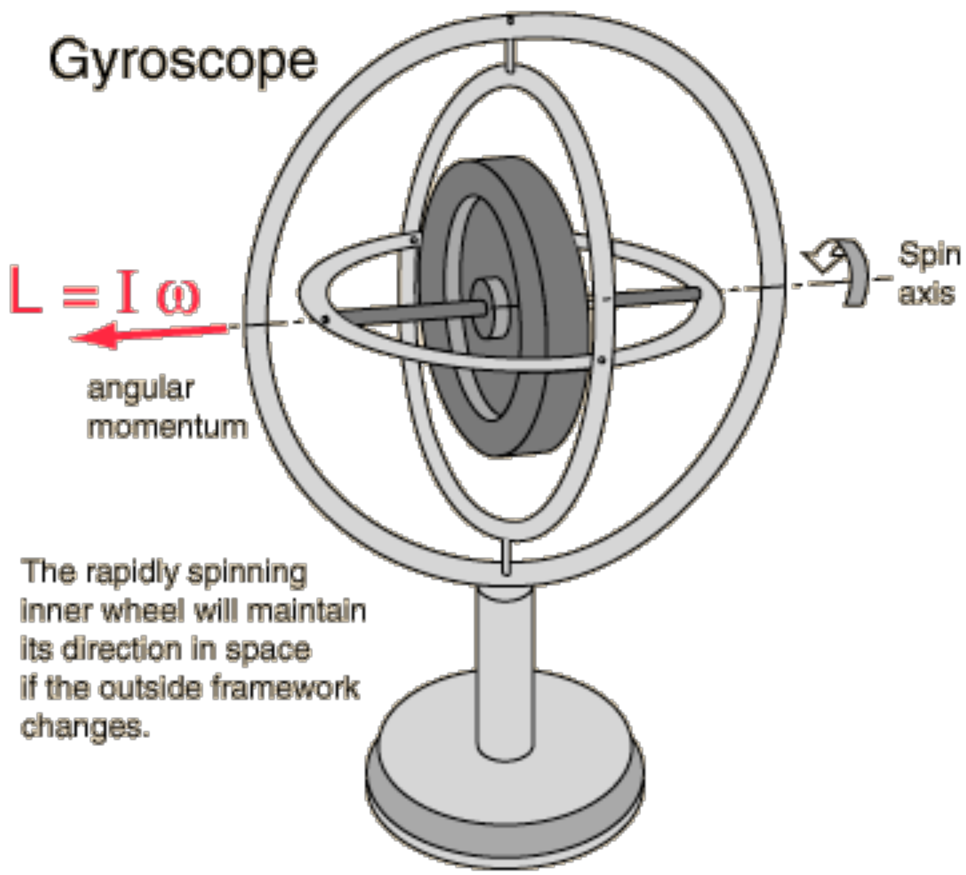
章动





陀螺仪 - 惯性制导系统



http://upload.wikimedia.org/wikipedia/commons/d/d5/Gyroscope_operation.gif

欧拉转盘

Moffatt, H. K. *Nature* **404**, 833–834 (2000)

Prepared by Jiang Xiao

Euler's disk and its finite-time singularity

Air viscosity makes the rolling speed of a disk go up as its energy goes down.

It is a fact of common experience that if a circular disk (for example, a penny) is spun upon a table, then ultimately it comes to rest quite abruptly, the final stage of motion being characterized by a shudder and a whirring sound of rapidly increasing frequency. As the disk rolls on its rim, the point P of rolling contact describes a circle with angular velocity Ω. In the classical (non-dissipative) theory[1], Ω is constant and the motion persists forever, in stark conflict with observation. Here I show that viscous dissipation in the thin layer of air between the disk and the table is sufficient to account for the observed abruptness of the settling process, during which, paradoxically, Ω increases without limit. I analyse the nature of this 'finite-time singularity', and show how it must be resolved.

Let α be the angle between the plane of the disk and the table. In the classical description, and with the notation defined in Fig. 1, the points P and O are instantaneously at rest in the disk, and the motion is therefore instantaneously one of rotation about line PO with angular velocity ω, say. The angular momentum of the disk is therefore $\mathbf{h}=A\omega\mathbf{e}(t)$, where $A=\frac{1}{4}Ma^2$ is the moment of inertia of the disk of mass M about its diameter; $\mathbf{e}(t)$ is a unit vector in the direction PO; $\mathbf{e}_z$, $\mathbf{e}_d$ are unit vectors in the directions Oz, OC, respectively (see Fig. 1). In a frame of reference rotating with angular velocity $\boldsymbol{\Omega}_d=\Omega\mathbf{e}_z$, the disk rotates about its axis OC with angular velocity $\boldsymbol{\Omega}_d=\Omega_d\mathbf{e}_d$; hence the rolling condition is $\Omega_d=\Omega\cos\alpha$. The absolute angular velocity of the disk is thus $\boldsymbol{\omega}=\Omega(\mathbf{e}_d\cos\alpha-\mathbf{e}_z)$, and so

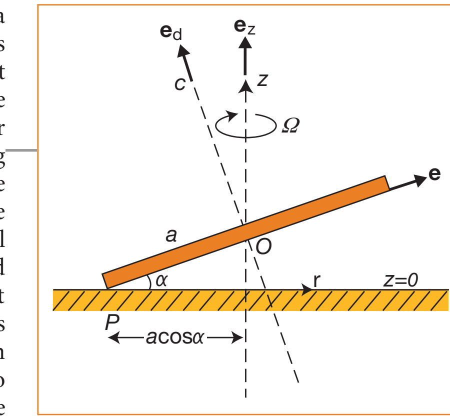


Figure 1 A heavy disk rolls on a horizontal table. The point of rolling contact P moves on a circle with angular velocity Ω. Owing to dissipative effects, the angle α decreases to zero within a finite time and Ω increases in proportion to $\alpha^{-1/2}$.

$\omega=\boldsymbol{\omega}\cdot\mathbf{e}=-\Omega\sin\alpha$.

Euler's equation for the motion of a rigid body is here given by $d\mathbf{h}/dt=\boldsymbol{\Omega}\wedge\mathbf{h}=\mathbf{G}$, where $\mathbf{G}=Mga\mathbf{e}_z\wedge\mathbf{e}$ is the gravitational torque relative to P ($\wedge$ indicates the vector product). This immediately gives the result $\Omega^2\sin\alpha=4g/a$, or, when α is small,

$$\Omega^2\alpha\approx 4g/a \qquad (1)$$

The energy of the motion E is the sum of the kinetic energy $\frac{1}{2}A\omega^2=\frac{1}{2}Mga\sin\alpha$, and the potential energy $Mga\sin\alpha$, so

$$E=\tfrac{3}{2}Mga\sin\alpha\approx\tfrac{3}{2}Mga\alpha \qquad (2)$$

In the classical theory, α, Ω and E are all constant, and the motion continues indefinitely. As observed above, this is utterly unrealistic.

Let us then consider one of the obvious mechanisms of energy dissipation, namely that associated with the viscosity μ of the surrounding air. When α is small, the dominant contribution to the viscous dissipation comes from the layer of air between the disk and the table, which is subjected to strong shear when Ω is large. We may estimate the rate of dissipation of energy in this layer as follows. Let (r,θ) be polar coordinates with origin at O. For small α, the gap $h(r,\theta,t)$ between the disk and the table is given by $h(r,\theta,t)\approx\alpha(a+r\cos\phi)$, where $\phi=\theta-\Omega t$. We now concede that α is a slowly varying function of time t: we assume that $|\dot\alpha|\ll\Omega$, and make the 'adiabatic' assumption that equation (1) continues to hold. Because the air moves a distance of order a in a time $2\pi/\Omega$, the horizontal velocity $\mathbf{u}_H$ in the layer has order of magnitude $r\Omega\sin\phi$; and as this velocity satisfies the no-slip condition on $z=0$ and on $z=h$ $(=O(\alpha a))$, the vertical shear $|\partial\mathbf{u}_H/\partial z|$ is of the order $(r\Omega/\alpha a)|\sin\phi|$. The rate of viscous dissipation of energy Φ is given by integrating $\mu(\partial\mathbf{u}_H/\partial z)^2$ over the volume V of the layer of air: this easily gives $\Phi\approx\pi\mu g a^2/\alpha^2$, using equation (1). The fact that $\Phi\to\infty$ as $\alpha\to 0$ should be noted.

The energy E now satisfies $dE/dt=-\Phi$ (neglecting all other dissipation mechanisms). Hence, with E given by equation (2), it follows that

$$\tfrac{3}{2}Mga\,d\alpha/dt\approx-\pi\mu g a^2/\alpha^2 \qquad (3)$$

This integrates to give

$$\alpha^3=2\pi(t_0-t)/t_1 \qquad (4)$$

where $t_1=M/\mu a$, and t_0 is a constant of integration determined by the initial condition: if $\alpha=\alpha_0$ when $t=0$, then $t_0=(\alpha_0^3/2\pi)t_1$. What is striking here is that, according to equation (4), α does indeed go to zero at the finite time $t=t_0$. The corresponding behaviour of Ω is $\Omega\approx(t_0-t)^{-1/6}$, which is certainly singular as $t\to t_0$.

Of course, such a singularity cannot be realized in practice: nature abhors a singularity, and some physical effect must intervene to prevent its occurrence. Here it is not difficult to identify this effect: the vertical acceleration $|\ddot h|=|a\ddot\alpha|$ cannot exceed g in magnitude (as the normal reaction at P must remain positive). From equation (4), this implies that the above theory breaks down at a time τ before t_0, where

$$\tau=t_0-t\approx(2a/9g)^{3/5}(2\pi/t_1)^{1/5} \qquad (5)$$

A toy, appropriately called Euler's disk[2], is commercially available (Fig. 2; Tangent Toys, Sausalito, California). For this disk, $M=400$ g, and $a=3.75$ cm. With these values and with $\mu=1.78\times10^{-4}$ g cm^{-1} s, $t_1=M/\mu a\approx 0.8\times10^6$ s, and, if we take $\alpha_0=0.1$ $(\approx 6°)$, we find $t_0\approx 100$ s. This is indeed the order of magnitude (to within $\pm20\%$) of the observed settling time in many repetitions of the spinning of the disk (with quite variable and ill-controlled initial conditions), that is, there is no doubt that dissipation associated with air friction is sufficient to account for the observed behaviour. The value of τ given by equation (5) is 10^{-2} s for the disk values given above; that is, the behaviour described by equation (4) persists until within 10^{-2} s of the singularity time t_0. At this stage, $\alpha\approx 0.5\times10^{-2}$, $h_0=a\alpha\approx 0.2$ mm, $\Omega\approx 500$ Hz (and the adiabatic approximation is still well

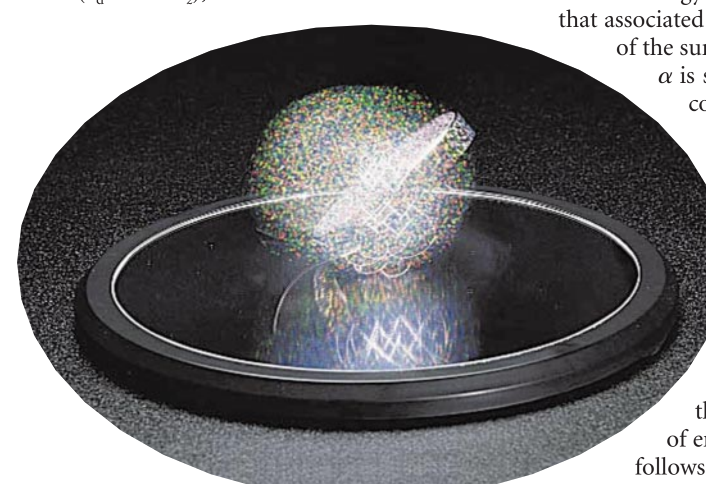
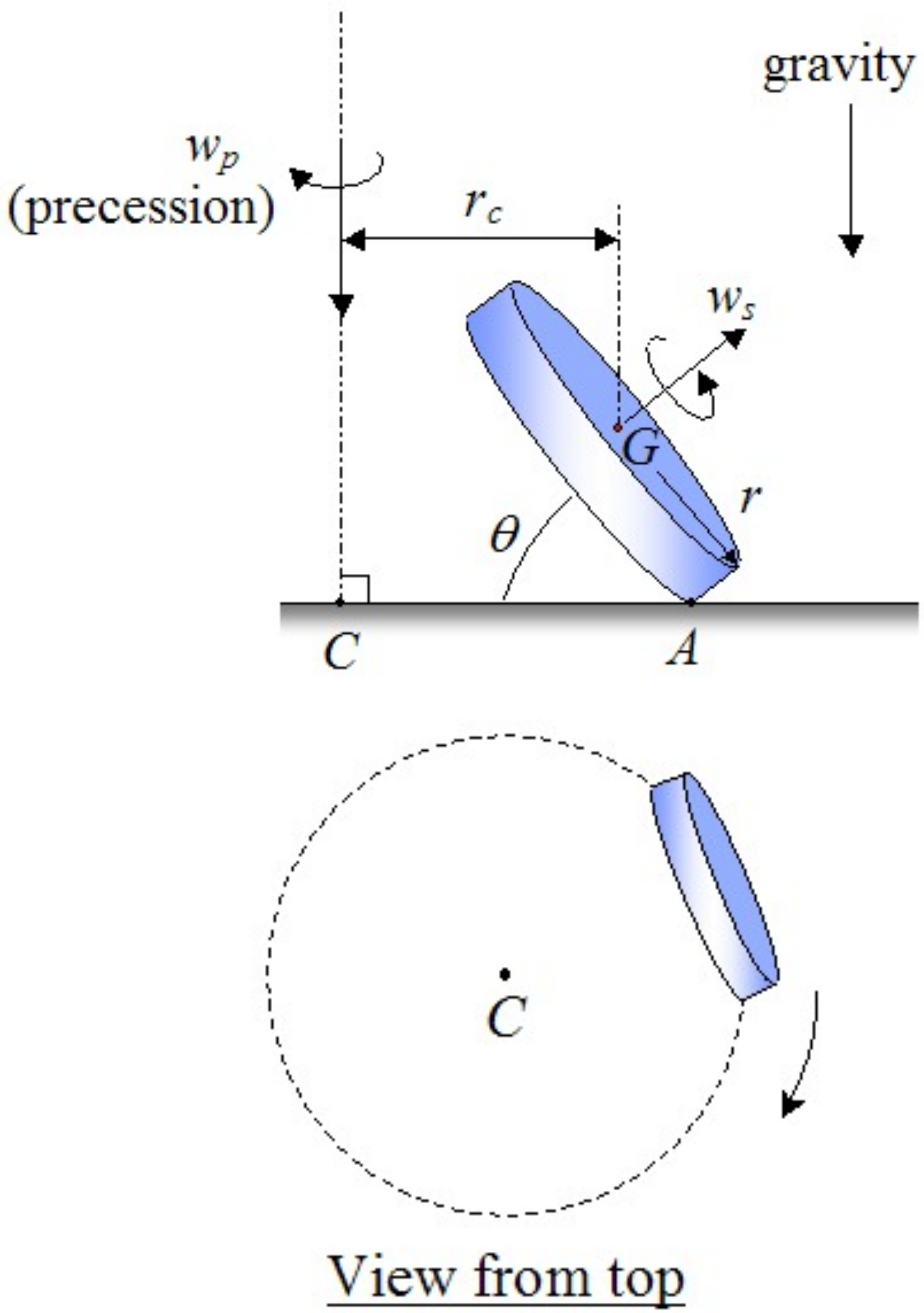


Figure 2 Euler's disk is a chrome-plated steel disk with one edge machined to a smooth radius. If it were not for friction and vibration, the disk would spin for ever. Photo courtesy of Tangent Toys. See http://www.tangenttoy.com/.

欧拉转盘



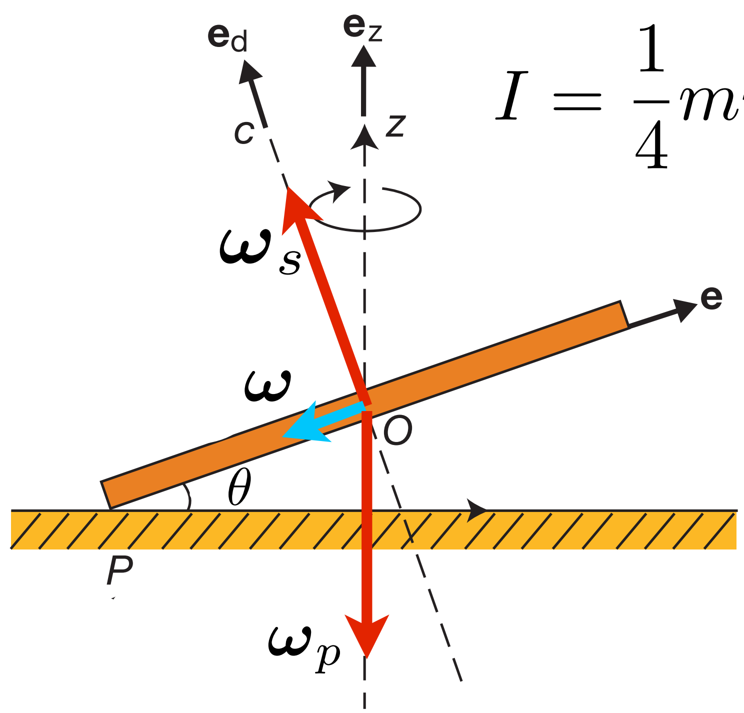
$$\boldsymbol{\omega} = \boldsymbol{\omega}_s + \boldsymbol{\omega}_p$$

$$(r_c + r\cos\theta)\omega_p = r\omega_s$$

$$\theta \to 0 \quad \Rightarrow \quad r_c \to 0$$

$$\omega_p \cos\theta = \omega_s$$

欧拉转盘



$$I = \frac{1}{4}mr^2$$

$$\boldsymbol{\omega} = \boldsymbol{\omega}_s + \boldsymbol{\omega}_p$$

$$\omega_p \cos\theta = \omega_s$$

$$\boldsymbol{\omega} = -\omega_p \sin\theta \hat{\mathbf{e}}(t)$$

$$\mathbf{L} = I\boldsymbol{\omega} = -I\omega\ \hat{\mathbf{e}}(t)$$

$$\mathbf{T} = -mgr\cos\theta\hat{\mathbf{e}}_\phi \quad = \quad \frac{d\mathbf{L}}{dt} = -I\omega\frac{d\hat{\mathbf{e}}(t)}{dt} = -I\omega\omega_p\hat{\mathbf{e}}_\phi$$

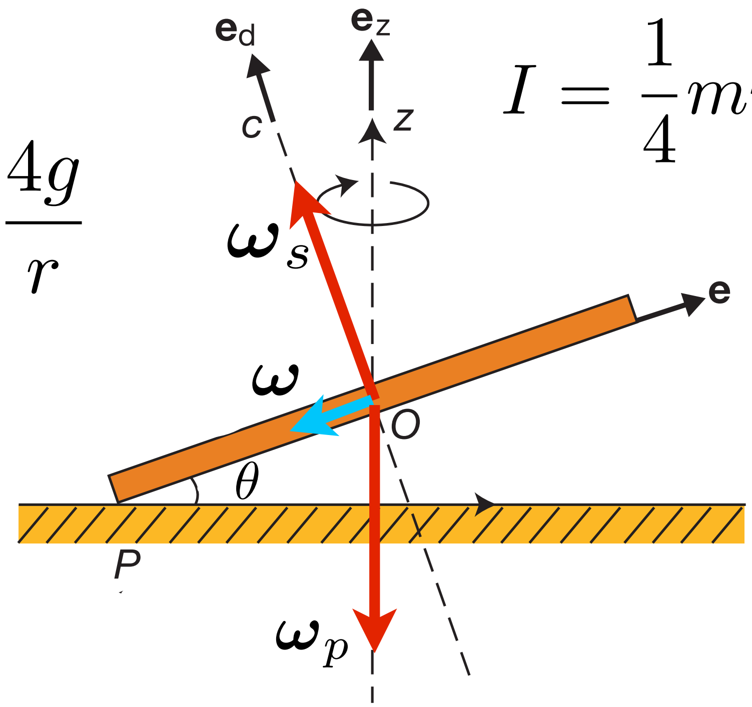
$$I\omega_p^2 \sin\theta = mgr\cos\theta \quad \Rightarrow \quad \omega_p^2\theta \approx \frac{4g}{r}$$

$$E_k = \frac{1}{2}I\omega^2 = \frac{1}{2}I\omega_p^2\sin^2\theta \approx \frac{mgr}{2}\theta \quad \text{and} \quad E_p = mgr\sin\theta \approx mgr\theta$$

$$E = E_k + E_p \approx \frac{3}{2}mgr\theta$$

欧拉转盘 - 能量耗散

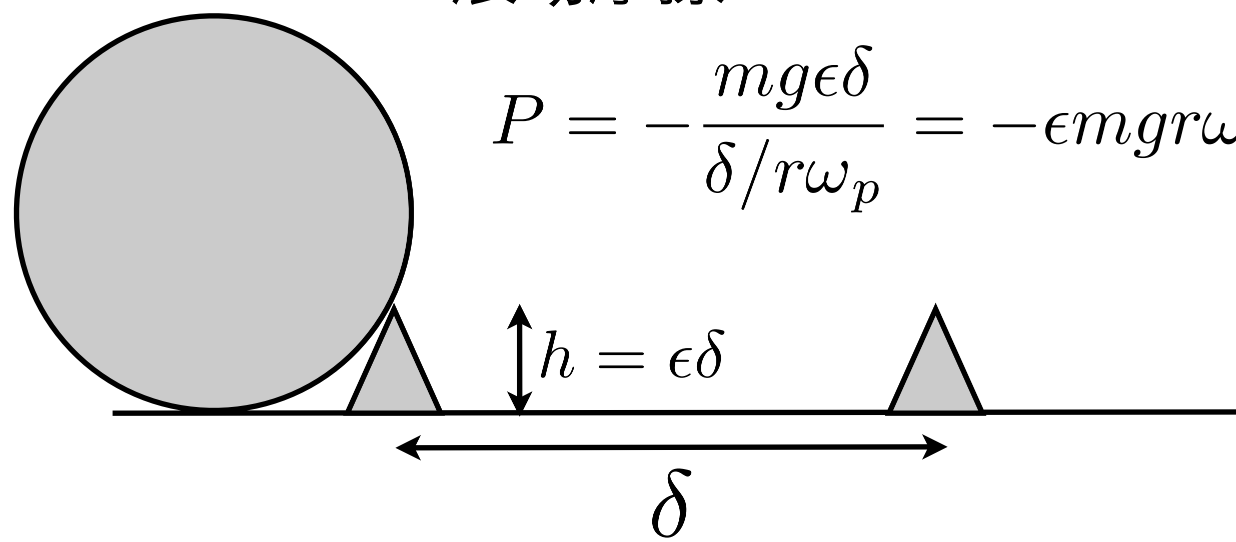
$$\omega_p^2\theta \approx \frac{4g}{r}$$



$$I = \frac{1}{4}mr^2$$

滚动摩擦：

$$P = -\frac{mg\epsilon\delta}{\delta/r\omega_p} = -\epsilon mgr\omega_p$$



$$E = E_k + E_p \approx \frac{3}{2}mgr\theta$$

$$\frac{dE}{dt} = P$$

滚动摩擦、空气

$$\frac{3}{2}mgr\dot{\theta} = -\epsilon mgr\sqrt{\frac{4g}{r\theta}} \quad \Rightarrow \quad \theta^{\frac{1}{2}}\dot{\theta} = \frac{d}{dt}\theta^{\frac{3}{2}} = -\frac{4}{3}\epsilon\sqrt{\frac{g}{r}}$$

$$\theta = \left[\frac{4}{3}\epsilon\sqrt{\frac{g}{r}}(t_0 - t)\right]^{\frac{2}{3}} \quad \Rightarrow \quad \omega_p \approx \frac{4g}{r\theta} \propto (t_0 - t)^{\frac{1}{3}}$$

Prepared by Jiang Xiao

第七章作业

习题：7.10, 7.16, 7.17, 7.19, 7.26

4月18日（周三习题课）交