Convective Cloak in Hele-Shaw Cells with Bilayer Structures: Hiding Objects from Heat and Fluid Motion Simultaneously

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Thermal convection is one of the three basic mechanisms of heat transfer, which profoundly influences the natural environment, social production, and our daily life. However, the high complexity of the governing equation, which describes the coupling of heat and mass transfer, makes it difficult to manipulate thermal convection at will in both theory and experiment. Here, we consider the heat transfer in Hele-Shaw cells, a widely used model of Stokes flow between two parallel plates with a small gap, and apply the scattering-cancellation technology to construct convective thermal materials with bilayer structures and homogeneous isotropic materials. By tailoring thermal conductivities and viscosities, we demonstrate cloaking devices that can hide obstacles from heat and fluid motion simultaneously, and verify their robustness under various thermal-convection environments by numerical simulations. Our results show that about 80% of the temperature and pressure disturbances in the background caused by obstacles can be eliminated by the cloak. The developed approach can be extended to control other convection-diffusion systems or even other multiphysics processes, and the results pave a promising path for designing various metadevices such as concentrators or sensors.

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I. INTRODUCTION

Metamaterials [1] (and metadevices [2]), usually made of artificially structured composites, have been a powerful tool to manipulate physical fields in many realms [1–4], and provide functions beyond naturally occurring materials. One typical methodology to design metamaterials is the transformation optics [5] and its counterparts in other physical fields [3]. However, the requirement for inhomogeneous and anisotropic parameters makes it difficult to fabricate devices designed by transformation optics. To overcome this bottleneck, scattering-cancellation technology (SCT) has been developed and successfully used in electromagnetism [6] and other fields [7]. Generally speaking, SCT can realize a similar function to transformation optics, whereas it only needs bilayer or monolayer structures and homogeneous isotropic bulk materials. SCT is also called “solving the equation directly” sometimes whereas, in fact, the main procedure of this method is inversely finding the coefficients (material parameters) of the equation according to the required solution with analytical techniques. One type of equation that is often dealt with by SCT is the Laplacian-type equation, which is just a Laplace equation in a homogeneous isotropic medium. Laplacian-type equations can describe the magnetic scalar potential in static magnetic fields [8], the temperature or electrostatic potential in heat or electrical conduction [9–14], or the liquid pressure in a potential flow [15,16]. On the basis of them, various bilayer or monolayer metamaterials have been realized in the mentioned scenarios and independent multiphysics [17–20].

However, although thermal convection is one of the basic modes of heat transfer and plays an important role in nature and human society, effective techniques such as SCT for manipulating it are still lacking. This dilemma may result from the complexity of its multiphysical governing equations. In detail, convective heat flux contains not only an advective term due to the movement of fluid medium but also a conductive term in the nonisothermal flow. Therefore, the governing equations consist of the conduction-advection heat equations, the law of continuity for fluid motion, and the Navier-Stokes equations, which, especially the Navier-Stokes equations, make it difficult to apply transformation optics or SCT. As a result, although

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thermal metamaterials [21,22] have been developed for more than a decade and show potential in practical applications such as thermal management of electronic devices, thermal camouflage imaging, and radiative cooling [23–25], the progress of metamaterials in thermal convection seems insufficient.

Current advances in convective thermal metamaterials mainly benefit from choosing an appropriate simplified model of the Navier-Stokes equations, for example, Darcy’s law which describes the creeping flow or Stokes flow (Reynolds number $Re \ll 1$) in porous media [26]. By engineering the permeability of porous media, some fluid-flow metamaterials have been designed [27,28], and this technique has been combined with the method of tailoring thermal conductivity to design convective thermal metamaterials [29–33]. In theory, convective thermal metamaterials can control heat flux and flow field simultaneously [29,30]. However, due to the limited practical means to tailor the permeability, reports on experimentally realizing such fluid-flow or convective metamaterials are still scarce. More recently, another hydrodynamic model has been used to control fluid motion, i.e., the Stokes flow inside two parallel plates, and a series of experimental works have been reported [34–37]. The gap between two plates is much smaller than the characteristic length of the other two spatial dimensions, so the model is also called the Hele-Shaw flow or Hele-Shaw cell [38]. As an extension of the Poiseuille flow [39], the Hele-Shaw flow is quite a fundamental model in many fields such as viscous fingering [40], microfluidics [41], parametric resonance [42], and flow-induced choking [43]. The fluid pressure in the Hele-Shaw cell also satisfies a Laplacian-type equation [44], and SCT has been employed to construct a monolayer fluid-flow cloak in the cell [15].

In this work, we develop SCT to control thermal convection in a Hele-Shaw cell and employ it for seeking suitable thermal conductivities and viscosities of artificial structures. That is, by surrounding an area with two layers of homogeneous isotropic material, we can obtain the desired temperature and pressure distribution (and, thus, heat flow and velocity distribution) inside and outside these two layers. Unlike previous works on SCT, which dealt with single or decoupled fields, we investigate how to apply SCT to a set of coupled equations and simultaneously regulate multiphysical fields. As an application, we design a bilayer convective (thermo-hydrodynamic) cloak which can prevent obstacles from disturbing the external thermal and flow fields simultaneously. We show how the respective cloaking conditions of heat conduction and fluid motion are combined and successfully work together in thermal convection. Our design is further verified by numerical simulation under various convective environments, showing high robustness.

**II. THEORY**

**A. Governing equations**

Given that heat transfers within a Hele-Shaw cell, in which the fluid demonstrates a creeping flow, the governing equations of this model contain the heat transfer equation [45]

$$\nabla \cdot (-\kappa \nabla T + \rho C^p T \nu) = 0, \quad (1)$$

the law of continuity for fluid motion

$$\nabla \cdot (\rho \nu) = 0, \quad (2)$$

and the Hele-Shaw equation [44]

$$\nu = -\frac{h^2}{12\mu} \nabla P, \quad (3)$$

which is a simplification of the Navier-Stokes equation. Here, $T$ is the temperature and $\nu$ is the velocity of fluid motion. We use $\kappa$, $\rho$, $C^p$, and $\mu$ to denote the thermal conductivity, density, specific heat at constant pressure, and the dynamic viscosity of the fluid, respectively. In addition, $h$ is the depth of the Hele-Shaw cell and $P$ is the fluid pressure. Strictly speaking, Eq. (3) gives the average velocity $\nu(x,y)$ along the $z$-axis if the plates of cells are put on the $x$-$y$ plane, so we can treat the three-dimensional (3D) model as a two-dimensional (2D) one. Applying the divergence operator on Eq. (3) and comparing it with Eq. (2), we can see

$$\nabla \cdot \left( \frac{\rho h^2}{12\mu} \nabla P \right) = 0. \quad (4)$$

In a region where $\rho$, $\mu$, and $h$ are all constants (or the ratio $\rho h^2/\mu$ remains unchanged), Eq. (4) is just Laplace’s equation and, thus, the Hele-Shaw flow is a classical potential flow such as the Darcy flow in the porous media. On the other hand, substituting Eq. (3) into Eq. (1) to eliminate the velocity term, the heat transfer equation can be written as

$$\nabla \cdot \left( \kappa \nabla T + \frac{\rho C^p h^2 T}{12\mu} \nabla P \right) = 0. \quad (5)$$

When the velocity is zero everywhere, Eq. (5) is also a Laplacian-type equation known as the Fourier’s law of heat conduction. As we mentioned previously, various bilayer metamaterials have been realized in pure heat conduction or potential flows. Our aim is to derive the material parameters to realize the similar functions as conduction under the convective environment.
B. Bilayer device model and SCT

Now we consider the case where both the thermal bias and the pressure bias are applied on the $x$ direction (see the heat sources and pressures applied in Fig. 1; other boundary conditions are discussed with simulation validation in Part C of the Appendix), and assume that

$$\nabla P = f(\mathbf{r})\nabla T$$  \hspace{1cm} (6)

in the whole space. Moreover, by performing a variable substitution $\varphi(\mathbf{r}) = f(\mathbf{r})(\rho C_p h^2/2\mu\kappa)$, Eq. (5) can be rewritten as

$$\nabla \cdot (\kappa (\nabla T + \varphi(\mathbf{r}) \nabla T)) = 0.$$  \hspace{1cm} (7)

If $\varphi$ is a constant in each domain, we can write the heat transfer equation with a compact form as

$$\nabla \cdot \left(\kappa \left(\nabla \left(T + \frac{1}{2}\varphi T^2\right)\right)\right) = \nabla \cdot (\kappa \nabla \Phi) = 0.$$  \hspace{1cm} (8)

Here $\Phi = T + (1/2)\varphi T^2$. This trick (similar to the Kirchhoff transformation in nonlinear heat conduction [46] which has been used in designing nonlinear thermal metamaterials [47]) makes the governing equation conform to the form of Laplace’s equation if $\kappa$ is also a constant. In the framework of bilayer metamaterials, we use subscripts 1, 2, 3, and $b$ to represent the central functional area inside the device, the inner layer, the outer layer, and the background outside the device, respectively [see Fig. 1(b)]. The radius from inside to outside [corresponding to core, inner, and outer layers in Fig. 1(a)] is $R_1$, $R_2$, and $R_3$.

SCT can be generalized as inverse analytical calculation of possible material parameters in each region according to the desired physical field distribution in some certain region. However, whether we want to modulate temperature or pressure distribution in a thermal convection environment, these two variables ($T$ and $p$) do not appear directly in the Laplace-like Eq. (8), but instead another variable $\Phi$. This is a key difference between coupled multiphysics and single physics. Fortunately, under certain conditions [see Eq. (A2) and the details of derivation in Part A of the Appendix], bilayer devices such as invisibility cloaks can be realized by simultaneously modulating thermal conductivity and viscosity. This cloak works in both thermal and hydromechanical fields. In the following, we deduce the design parameters of such a cloak.

C. Condition for a convective cloak

As generally defined in metamaterials, cloaking can realize invisibility [5]. It means the scattering signals from an obstacle can be eliminated by a specific device surrounding it [named Criterion I which requires $T_b(r; r > R_3) = T_{\text{Ref}}(r; r > R_3)$ in heat transfer and the flux cannot flow into the obstacle (named Criterion II which requires $c = 0$). For Laplacian-type governing equations with diffusion nature, scattering signals mean distortion of potential (such as temperature, fluid pressure, and electrostatic potential) distribution in the background. As Eq. (8) and Fourier’s law have the same form for $\Phi$ and $T$, we can expect that the thermal conductivity for a thermal cloak in convection is also similar to its counterpart in conduction. Of course, as mentioned before, we need to note that the independent variables and boundary conditions of these two equations are different. Therefore, the physical meanings of the corresponding conclusions are not exactly the same, unless there is no advection. In heat conduction, the condition for Criterion I using the general anisotropic monolayer structure has already been solved [48], and its version for the isotropic bilayer scheme is

$$\kappa_1 = \frac{D_1 + D_2 + D_3 + D_4}{D_1 - D_2 - D_3 + D_4} \kappa_2,$$  \hspace{1cm} (9)

where $D_n (n = 1, 2, 3, 4)$ is given by

$$D_1 = - (\kappa_2 + \kappa_3) (\kappa_3 + \kappa_b) R^2_2,$$  \hspace{1cm} (10a)

$$D_2 = - (\kappa_2 + \kappa_3) (\kappa_3 + \kappa_b) R^2_1.$$  \hspace{1cm} (10b)

![FIG. 1. Schematic design for a bilayer convective cloak in a Hele-Shaw cell. (a) Side view of the cell. (b) Top view of this quasi-two-dimensional model (in the x-y plane).](image-url)
\[ D_3 = (\kappa_2 + \kappa_3) (\kappa_3 - \kappa_b) R_3^2, \quad (10c) \]
\[ D_4 = (-\kappa_2 + \kappa_3)(\kappa_3 - \kappa_b) \frac{R_1^2 R_2^2}{R_2^2}. \quad (10d) \]

In addition, previous works on preventing heat flux from entering into the cloak usually require that the inner layer is absolutely insulated, meaning that \( \kappa_2 = 0 \) [9–11]. In this case, \( \kappa_1 \) can take arbitrary real values, so the denominator in Eq. (9) must be zero, which can result in the familiar relationship [11]

\[ \kappa_3 = \frac{R_3^2 + R_2^2}{R_3^2 - R_2^2} \kappa_b. \quad (11) \]

We must emphasize that, by now, Eq. (11) can only be seen as the condition for shielding scattering signals of \( \Phi \) in thermal convection. Actually, thermal conductivity engineering alone is not enough to achieve a cloak (for \( \Phi \)) in convection as our assumption on \( f \) or \( \varphi \) actually requires a certain viscosity distribution. From the condition that \( \varphi \) is a constant [see Eq. (A3) in Part A of the Appendix], we should have

\[ \mu_2 = \infty, \quad \mu_3^{-1} = \frac{R_3^2 + R_2^2}{R_3^2 - R_2^2} \mu_2^{-1}. \quad (12) \]

It is interesting that Eq. (12) is exactly the condition for a bilayer fluid-flow cloak [15,16].

So far, we actually make a cloak for the potential \( \Phi \) and pressure \( P \). We should still verify the two criteria for \( T \). First, from \( \Phi_b = \Phi_{\text{Ref}}, P_b = P_{\text{Ref}} \), and the fact that the materials in reference and background are the same, we can obtain a differential equation for \( T_b - T_{\text{Ref}} \) as

\[ \nabla (T_b - T_{\text{Ref}}) = (T_{\text{Ref}} - T_b) \frac{\rho_b C_{\text{Ph}}^b h_b^2}{12 \mu_b \kappa_b} \nabla P_b. \quad (13) \]

In a homogeneous medium, \( \nabla P_{\text{Ref}} \) is a uniform field along the \( x \) axis, so there are two general solutions for Eq. (13), namely \( T_b - T_{\text{Ref}} \equiv 0 \) and \( T_b - T_{\text{Ref}} \sim \exp \left( \frac{-\nabla x P_b h_b^2}{12 \mu_b \kappa_b} \right) \). As \( T_b - T_{\text{Ref}} \) must vanish on the left and right boundaries, the only possible solution is the trivial one, so Criterion 1 is met. Similarly, it can be deduced from \( \nabla \Phi_1 = 0 \) and \( \nabla P_1 = 0 \) that \( \nabla T_1 = 0 \). In conclusion, by tailoring thermal conductivity and viscosity based on Eqs. (11) and (12), the target objects can be hidden in heat and fluid fields simultaneously within the artificial structures.

It should be emphasized that, in our derivation, we do not give the assumption of whether the obstacle is a solid object or not. Under ideal conditions, the obstacle (and the inner layer) cannot move so this concern does not matter. In fact, a perfect cloak (\( \kappa_2 = 0 \) and \( \mu_2 = \infty \)) does not care about the material inside because the inner layer isolates internal and external interactions. An interesting argument is that, if the material inside the cloak has an extremely low conductivity (\( \kappa_1 \to 0 \)) and high viscosity (\( \mu_1 \to \infty \)), Criterion II is met. Otherwise, an imperfect inner layer (with small and positive \( \kappa_2 \) and \( 1/\mu_2 \)) can be regarded as approximately insulated and immobile as long as

\[ \frac{\mu_2}{\mu_3} \ll 1, \quad \frac{\mu_2}{\mu_3} \ll 1, \quad \frac{\kappa_2}{\kappa_3} \ll 1, \quad \frac{\kappa_2}{\kappa_b} \ll 1. \quad (14) \]

In particular, when the inner layer and the central area are occupied by the same material, the bilayer cloak degenerates to a monolayer cloak.

### III. SIMULATION RESULTS

Now we verify our theoretical design by numerical simulation with the help of commercial finite-element software COMSOL MULTIPHYSICS (https://www.comsol.com/). As depicted in Fig. 1, the depth of 2D cell model is an extra parameter in the creeping flow module. In fact, in addition to the law of continuity, the governing equation of the Hele-Shaw flow or shallow channel approximation in COMSOL MULTIPHYSICS is \( \nabla P - \nabla (\mu (\nabla v + \nabla v^T)) + \frac{12 \mu}{h^2} v = 0 \). When \( h \to 0 \), this equation is reduced to Eq. (4). An alternative method is using the mathematics module to establish and solve Eq. (4). For the background material, we use water as a reference and set \( \kappa_b = 0.6 \text{ W m}^{-1} \text{ K}^{-1}, \mu_b = 10^{-3} \text{ Pa s}, \rho_b = 1000 \text{ kg m}^{-3}, \) and \( C_{\text{Ph}}^b = 5000 \text{ J kg}^{-1} \text{ K}^{-1} \). The depth of the cell is set as \( h = 2 \times 10^{-6} \text{ m}, \) and the horizontal section (the \( x-y \) plane) is a square with side length equal to \( 2 \times 10^{-4} \text{ m} \). The radius of the central region, inner layer and outer layer are respectively \( R_1 = 0.25 \times 10^{-4} \text{ m}, R_2 = 0.4 \times 10^{-4} \text{ m}, \) and \( R_3 = 0.5 \times 10^{-4} \text{ m} \). In addition, the depth of the cell, the specific heat, and the density are set the same everywhere for the uniform product \( \rho C_{\text{Ph}} h^2 \). The applied temperature bias and pressure difference are 10 K and 500 Pa, respectively. The hot source (303.15 K) and the fluid inlet are set on the left boundary of the whole system whereas the cold source (293.15 K) and the fluid outlet are both set on the right side. On the upper and bottom boundaries, we apply thermal insulation and nonslip conditions. If we do not consider the boundary layer, based on Eq. (3), the flow speed in the reference is \( 8.3 \times 10^{-3} \text{ m/s} \). Using \( 2h_0 \) as the characteristic linear dimension, the Reynolds number for the reference is \( 1.7 \times 10^{-4} \), which is consistent with the creeping flow hypothesis. For the cloak, we cannot achieve the perfect infinite thermal conductivity and zero viscosity in numerical simulation. Nevertheless, based on Eq. (14), we can set the inner layer approximately nonconductive and motionless with \( \kappa_2 = 6 \times 10^{-4} \text{ W m}^{-1} \text{ K}^{-1} \) and \( \mu_2 = 100 \text{ Pa s} \). The parameters of the outer layer are \( \kappa_3 = 2.73 \text{ W m}^{-1} \text{ K}^{-1} \) and \( \mu_3 = 2.20 \times 10^{-4} \text{ Pa s} \). Inside
the convective cloak, we take $\kappa_1 = 2.4 \text{ W m}^{-1} \text{ K}^{-1}$ and $\mu_1 = 0.01 \text{ Pa s}$ for the fluid material obstacle.

The simulation results of steady temperature and pressure performances are shown in Figs. 2(a1)–2(b3). In addition to the cloak, “Reference” represents the bare scenario with neither the cloak nor the obstacle, whereas “Without cloak” scenario means putting only the obstacle into “Reference.” According to isotherms and isobars, the temperature and pressure distribution of the convective cloak in the background are basically the same as the reference, and the temperature and pressure inside the cloak is visually invariant in space. As a comparison, the obstacle distorts isotherms and isobars obviously. In particular, for intuitive comparison, we detect the data from the line segment $y = 0 \ (-10^{-4} \text{ m} < x < 10^{-4} \text{ m})$ in Figs. 2(a1)–2(b3) and plot them in Figs. 2(c) and 2(d) for the temperature and pressure.

**FIG. 2.** Simulation results of a convective cloak and its contrast. (a1)–(a3) Temperature distribution of the cloak, reference, and the case without a cloak, respectively, with 20 white isotherms. (b1)–(b3) Corresponding pressure distributions, also with 20 white isobars. (c), (d) Comparison of the temperature and pressure particularly in a chosen line segment $y = 0$ with data taken from (a1)–(b3). The gray regions with different shades represent the bilayer structure ($-0.5 \times 10^{-4} \text{ m} < x < -0.25 \times 10^{-4} \text{ m}$ and $0.25 \times 10^{-4} \text{ m} < x < 0.5 \times 10^{-4} \text{ m}$) or the central obstacle ($-0.25 \times 10^{-4} \text{ m} < x < -0.25 \times 10^{-4} \text{ m}$) of the cloak.
pressure, respectively. In both Figs. 2(c) and 2(d), the blue dashed line (“Cloak”) and black solid line (“Reference”) match well in the background. Inside the cloak, the temperature and pressure patterns for the cloak show plateaus, which indicates the heat flux is almost zero and the obstacle region approximately demonstrates no fluid flow. Further, we can define a ratio measuring the extent to which the disturbance of the external background by the obstacle is eliminated. The average absolute value of the temperature difference in the background area between the cloak and the reference is 0.016 K. Its counterpart between the reference and the case without a cloak is 0.095 K. Thus, 82% ($\approx 1 - 0.016/0.095$) of the temperature disturbance generated by the obstacle can be eliminated. The similar ratio for pressure can also be calculated as 77%. If we replace the fluid material obstacle with a solid one, the cloaking function for heat transfer and fluid flow still works well although the fluid pressure inside the cloak is absent for a solid. In addition, to make a perfect bilayer cloak with extreme parameters in numerical simulation, we can set the thermal insulation condition and the nonslip condition on the boundaries of the inner layer. However, we should note now the thermal and hydrodynamic fields in the central area (if it is still occupied by fluids) cannot be determined without giving extra information, although the background region would not be disturbed.

In Fig. 3, we show the heat flux density and velocity distributions for all the three convective devices and the reference. The advective heat flux density vector is defined as $\rho C_p(T - T_{Amb})v$, where the ambient temperature $T_{Amb}$ is 293.15 K. From Figs. 3(a)–3(d), it can be verified that the heat and mass fluxes are both blocked in the inner layer and the obstacle, and the fluxes outside the bilayer structure are the same as the reference. In addition, we can find Reynolds number in the background for the cloak is $Re \approx 2.07 \times 10^{-3}$ if we use $R_3$ as the characteristic scale. It is known that the boundary layer can play an important role in thermal convection. As the nonslip condition is applied on the upper and bottom boundaries (i.e., $y = \pm 10^{-4}$ m), the velocity there should be zero, so we plot the flux data taken from the line segment $x = 0$ ($-10^{-4}$ m < $y$ < $10^{-4}$ m) in Figs. 3(e) and 3(f). We can see the velocity increases or decreases sharply near $x = -0.5 \times 10^{-4}$ m or $x = 0.5 \times 10^{-4}$ m in Fig. 3(d), which corresponds to the laminar boundary layer. In addition, the heat flux exhibits sharp changes near the boundary, but to a lesser extent than the change in velocity. This can be explained by the Péclet number $Pe = Re \times Pr$ ($Pr$ is the Prandtl number).

![Fig. 3](image)

**FIG. 3.** Simulation results of (a),(b) heat flux density and (c),(d) velocity distributions for the cloak and the reference. The contour maps show the magnitude of heat flow or velocity vectors. The black arrows indicate the vector direction and the lengths of them also represent the vector size. (e), (f) Comparison of the heat flux density and speed particularly in a chosen line segment ($x = 0$) with data taken from (a)–(d).
For the reference, $Pe \approx 0.7$, so the conductive heat transport and the advective one are comparable in magnitude. Then, the advective heat flux demonstrating an obvious boundary layer plus the relatively uniform conductive heat flow results in the patterns in Fig. 3(f). For the same fluid material, the Prandtl number is fixed, so the boundary layer is actually affected by the Reynolds number. At the same time, according to our previous work [30], a larger Reynolds number also leads to a change in the pattern of the temperature distribution: the isotherm area is not uniform. Thus, we observe the performance of the cloak under different pressure differences in Part B of the Appendix and find our design works well.

To mimic a more realistic working environment, we perform a 3D simulation using the same material and structure parameters. Here, the shallow channel term $12\mu v/h^2$ added in the 2D Navier-Stokes equations is not needed. The inertia term is also included in the governing equations so we actually use the full 3D incompressible Navier-Stokes equations $\nabla p - \nabla \cdot (\mu (\nabla v + (\nabla v)^T)) + \rho (v \cdot \nabla)v = 0$. The simulation results are shown in Fig. 4. Here, because the velocity on the surface $z = \pm (1/2)h$ is zero, we should note that the temperature or pressure distribution on the surface (mainly referring to the planes $z = \pm (1/2)h$) is different from that on the central plane $z = 0$, and in 2D simulations it is actually the latter that is of interest. Anyway, from Fig. 4, we can see the distributions on the surface and the cut plane both show good cloaking effects in a low-speed flow environment.

### IV. DISCUSSION AND CONCLUSION

By now, the coupling of the thermal field and fluid movement is unidirectional. In other words, only the velocity would influence the temperature distribution because we have not considered the thermal response of fluid properties such as density and viscosity, which might be important in a nonisothermal flow. Taking water as the working medium as we have done in his article, the changing of density is not significant compared with that of viscosity under the applied thermal bias [49]. The dynamic viscosity of water can be expressed as a function of temperature with three parameters: $\mu = 10^{A+B/\Delta T-C}$ Pa s, where $A = -4.5318$, $B = 220.57$ K, and $C = 149.39$ K [50]. Taking $T = 20^\circ$C, we can see $\mu \approx 1 \times 10^{-3}$ Pa s, which is just the value we have used for the background material in simulation. When $\Delta T = 10$ K, the thermal response of $\mu_b$ still has little influence on the temperature or pressure distributions no matter whether we assume $\mu_1$, $\mu_2$, and $\mu_3$.

![Fig. 4. Simulation results using 3D Navier-Stokes equations: (a1),(b1) temperature and pressure distributions of the surface, respectively; (a2),(b2) distributions of the central $x$-$y$ plane ($z = 0$).](image-url)
change with temperature of the same magnitude or let them remain temperature-independent. When the thermal bias further increases, for example, to $50$ K, and the four viscosities involved have the same temperature dependence, the functions such as cloaking, sensing, and concentrating should not fail but the pressure distribution will demonstrate uneven isobars. This variable viscosity (in fact, its reciprocal) behaves like a nonlinear thermal conductivity in bilayer conductive metadevices [47] (we can do a power series expansion to $1/\mu$ and obtain the polynomial form of temperature just like the nonlinear thermal conductivity often used in research). On the other hand, the viscous dissipation term can also be neglected in the framework of creeping flows, compared with the convective heat transfer. The discussion above might help to improve the feasibility of our design in potential practical applications.

Another important question is how to make $\kappa$ and $\mu$ tunable in fluid materials. One idea is adding some inclusions or suspensions (such as nanoparticles and even active matter) into the medium [51–53]. However, we must prevent the inclusions from moving from one domain to another and changing the spatial distribution of $\kappa$ and $\mu$. In some recent research [34–36], solid pillars were put into the cell and fixed, which can enhance the effective viscosity of the solid-fluid structure. If viewed from another angle, this technique reduces the cell depth to zero without changing the viscosity of fluid itself. In fact, this technique also changes the thermal conductivity, specific heat, and density in the solid domain and, thus, influences the corresponding effective properties of the composites. Thus, the situation can be more complicated involving tuning $\kappa$, $\mu$, $\rho$, $C^\rho$, and even $h$. In this way, an effective medium theory considering heat transfer and fluid flow simultaneously is needed to inversely design suitable structures. In Part D of the Appendix, we give a 3D cloak structure with only one fluid material by changing the depth of the outer layer and putting pillars in it. Although the parameter estimation is empirically given and relatively rough, our design does exhibit some invisibility effect.

In summary, we have established a framework to design bilayer convective metamaterials in a Hele-Shaw cell through SCT. We extend this approach to deal with coupled multiphysics. By engineering thermal conductivity and viscosity, we have proposed a convective cloak that can realize thermal invisibility and hydrodynamic stealth at the same time. We also discuss the implications of the Reynolds number and directions of applied thermal bias and driving pressures, and the design shows high robustness under different convective circumstances. Although we only consider circular layers surrounding a rounded area, our design can be generalized to other geometries based on the existing and future research on Laplacian bilayer metamaterials, e.g., the elliptical structures [12,47]. The material parameters needed in our design for each layer and the central area are all homogeneous and isotropic, which could be achieved by solid-fluid composites, and the related effective medium theory or inverse design technique is to be further developed. Our study might provide a promising method for feasible and flexible control of multiphysics processes.

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**APPENDIX**

1. Details on SCT for thermal convection

For the model discussed in Sec. II (see Fig. 1), the general solution of $\Phi_i$ ($i = 1, 2, 3$, and $b$) in the 2D scenario with the circular symmetry can be expressed as

$$
\Phi_i = (A_{i\phi} + B_{i\phi} \ln r)(\alpha_{i\phi} + \beta_{i\phi}\theta) + \sum_{m=1}^{\infty} (A_{im}r^m + B_{im}r^{-m})(\alpha_{im}\sin(m\theta) + \beta_{im}\cos(m\theta))
$$

(A1)

with polar coordinates $(r, \theta)$. By finding the right coefficients $A_{im}, B_{im}, \alpha_{im}$ and $\beta_{im}$ based on the required manipulation function and certain boundary conditions, we can first obtain the inverse solution of thermal conductivities $\kappa_i$ to realize the manipulation of $\Phi$. Whether the required function can be realized for the temperature still needs some debate. Different from the familiar boundary conditions at infinity such as $\nabla, T(r = \infty) \sim \mathbf{e}_x$ ($\mathbf{e}_x$ is the unit vector along the $x$ axis), here we should use $\nabla, \Phi(r = \infty) \sim \mathbf{e}_x$ instead and, thus, the temperature of a homogeneous medium could not vary linearly along the $x$ axis. The degree of such nonuniformity depends on the value of $\varphi T$, and the advection part behaves just like a nonlinear thermal conductivity. In addition, to guarantee $\varphi$ is a constant in the whole system (thus, $T$ cannot be a multivalued function on the boundary of two domains), we must require

$$
f_j(\mathbf{r}) \frac{\rho_j C_{ij} \kappa_j^2 h_j^2}{\mu_j \kappa_j} = f_j(\mathbf{r}) \frac{\rho_j C_{ij} \kappa_j^2 h_j^2}{\mu_j \kappa_j} \equiv C, \quad i \text{ or } j = 1, 2, 3, b.
$$

(A2)
Here $C$ is a constant. Further, we can assume $f$, $\rho$, $C^p$ and $h$ are also constants in the whole space, and obtain

$$\frac{1}{\mu_1 \kappa_1} = \frac{1}{\mu_2 \kappa_2} = \frac{1}{\mu_3 \kappa_3} = \frac{1}{\mu_b \kappa_b} \equiv C', \quad (A3)$$

where $C'$ is another constant. In other words, we should tailor $\mu$ to realize a convective metamaterial in addition to thermal conductivity engineering. It should be noted that Eq. (A3) also gives the condition for a bilayer fluid-flow metamaterial, if we only consider Eqs. (2) and (3). In fact, we also need to neglect the difference between thermal insulation (the heat flow perpendicular to the boundary is zero) and nonslip (the velocity at the solid boundary is zero) boundary conditions. If the boundary layer is not significant, this neglect can be reasonable. For example, after calculating the thermal conductivities to avoid disturbing the distribution of $\Phi$ in the background, we can obtain the viscosities not to disturb $P$. Based on Eq. (4), tuning the ratio $\rho h^2/\mu$ is a more general strategy to make fluid-flow metamaterials. For simplicity, we only consider changing the conductivity and viscosity in the theory part. In this way, $f$ is assumed to be a constant and we must emphasize this is an approximation. When advection exits, $\mu$ actually cannot be a strict constant, e.g., in a homogeneous medium. Nevertheless, we can keep this assumption and check how much the variable $f$ will influence the performance of our design in the numerical simulation part.

In particular, for a convective cloak, we can find the assumption described by Eq. (A2) can be relaxed by only requiring that it is valid in the background and outer layer. As we discuss in Part C of Sec. II, the zero conductivity and infinite viscosity in the inner layer are sufficient to make Eq. (5) automatically satisfied in the inner layer and the obstacle. As a result, we do not let $\mu_1/\mu_b = \kappa_b/\kappa_1$ or $\mu_2/\mu_b = \kappa_b/\kappa_2$ for the numerical simulations in Sec. III. Furthermore, because the boundary layers in Fig. 3 are quite thin, it will not undermine our assumption Eq. (A2), i.e., the coefficient $f(r)$ in Eq. (6) should be a constant because the density, specific heat, and cell depth are all kept invariant.

2. Performance under different reynolds numbers

As we discuss in Sec. III, it is important to further verify our conclusions under different Reynolds numbers (or Péclet numbers). Compared with pure conduction ($Re = 0$), a large $Re$ can cause an obvious change of the

![FIG. 5. Simulation results under different Reynolds numbers: temperature distributions of the cloak and reference, respectively, when the pressure difference applied is (a1),(a2) $10^2$ Pa, (b1),(b2) $10^3$ Pa, or (c1),(c2) $10^4$ Pa.](image-url)
temperature patterns, which means \( f \) is not strictly a constant even for the reference. Nevertheless, we can still test the performance of the cloak designed here. We perform three more sets of simulations when \( \Delta P \) takes \( 10^2 \) Pa, \( 10^3 \) Pa, and \( 10^4 \) Pa, respectively, and show the results of the cloak and reference in Fig. 5. As the patterns of pressure distributions should not be changed under the Hele-Shaw regime, we only illustrate the temperature distributions here.

In Figs. 5(a1) and 5(a2), the pressure difference is \( 10^2 \) Pa, and the isotherms of the reference are almost evenly distributed. In Figs. 5(b1) and 5(b2), the pressure difference is \( 10^3 \) Pa, and the isotherms of the reference show a distinctly uneven distribution. When the advective heat transfer is further enlarged in Figs. 5(c1) and 5(d1), the conductive flux can be neglected. Then the isotherms would be crowded on the side of the cold source (although we did not draw these overly dense isotherms) and the flow is isothermal almost everywhere. Thus, the temperature gradient is close to zero except near the cold source. Anyway, we can see the cloaking effect is robust as well as the assumption of creeping flow is still valid in the Hele-Shaw cell. Here, the temperature disturbance caused by the obstacle is eliminated by 95%, 81%, and 79%, respectively, for the three columns in Fig. 5. We can see the cloaking ratio increases with the contribution of conductive heat transfer because our assumption that \( f \) is a constant is fulfilled perfectly when advection is absent.

3. Performance under nonparallel applied thermal and pressure biases

Now we turn to another aspect and consider the two applied biases are not in the same direction. Under this assumption, the relationship described by Eq. (6) should be revised. In general, we can decompose \( \nabla P \) into two components which are parallel (\( \nabla P_{\parallel} \)) and perpendicular (\( \nabla P_{\perp} \)) to \( \nabla T \), respectively, writing

\[
\nabla \cdot \left( \frac{\rho \gamma h^2 T}{12 \mu} \nabla P \right) = \frac{\rho \gamma h^2 T}{12 \mu} \nabla T \cdot \nabla P_{\parallel}. \tag{A4}
\]

Now \( f(\mathbf{r}) \) is defined by \( \nabla P_{\parallel} = f(\mathbf{r}) \nabla T \) and equal to \( (\nabla P \cdot \nabla T) / (\nabla T \cdot \nabla T) \). A special case is when the applied temperature difference and pressure difference are perpendicular to each other. In this case, we expect the patterns of temperature and pressure distributions approximately have a symmetry under rotating 90°. Then, \( f \) should be almost a constant (zero) [29] and we can apply SCT again and obtain the same parameters for the convective cloak.

To test our design, we let the pressure bias take 500 Pa along the \( y \) axis while the thermal bias is kept at 10 K along the \( x \) axis. In Figs. 6(a)–6(d) we give the simulation results of the temperature and pressure distributions for both the cloak and the reference. The isotherms and isobars show the cloaking effect is achieved. More specifically, Figs. 6(e) and 6(f) are on a horizontal and a vertical

\[\text{FIG. 6. (a),(b) Temperature or (c),(d) pressure distribution of a convective cloak and the reference when the applied thermal bias and the pressure bias are vertical. (e),(f) Temperature and pressure with data detected from the line } y = 0 \text{ and line } x = 0, \text{ respectively.}\]"
line segments. We can also calculate the percentage of disturbances that are removed in this situation. The ratio for pressure deviation should be the same as its counterpart under parallel applied thermal and pressure biases. The ratio for temperature deviation is a little bit different, taking 79%.

4. A 3D design for a convective cloak

Here we propose a 3D structure (see Fig. 7) to realize a convective cloak and use only one fluid material. The gray region in Fig. 7 illustrates the fluid domain. For simplicity, the inner layer of the cloak is provided as a solid thermal insulation material, so the structure is a monolayer one. In addition, we can see the outer layer has a larger depth than the background, which allows the fluid here to have a smaller viscosity. Furthermore, the larger depth of the fluid domain can enlarge the effective thermal conductivity due to a larger heat transfer cross-section. However, the increase in effective thermal conductivity caused by the depth alone does not exactly make the outer layer meet the conditions for a thermal cloak. More precisely, the thermal conductivity of the outer layer needs to be further improved. We here combine the two methods used in Refs. [15,34]. The outer layer is deeper than required for a normal fluid cloak in Ref. [15] and the too low viscosity is compensated for by putting pillars [34] [see the black dots placed in the outer layer in Figs. 7(a) and 7(b)] to realize to fluid cloak again. Then, we can tune the thermal conductivity (and the heat capacity) are not easy to solve through existing effective medium theory.

As a rough estimation, we still use the same background material (water) and take the depth of the outer layer as $\sqrt{R_3^2 + R_2^2/R_3^2 - R_2^2} \approx 2.13$ times the background (the ratio is $R_3^2 + R_2^2/R_3^2 - R_2^2$ in Ref. [15] for a fluid cloak). In addition, four rings of cylindrical pillars are placed in the outer layer, and each ring consists of 90 pillars with a radius of $\sqrt{f_p(R_3^2 - R_2^2)/N}$. Here $N = 360$ is the total number of pillars, and $f_p$ is the volume fraction of the pillars compared with the outer layer, which takes 0.32% in the following simulation. The material occupying the pillars can be air, soft matter, or solid. Here we set its thermal conductivity to be 40 W m$^{-1}$ K$^{-1}$, its density to be 1000 kg m$^{-3}$, and its specific heat to be 5000 J kg$^{-1}$ K$^{-1}$, which might be achieved by some mixture (e.g., copper and polydimethylsiloxane [54]). The simulation results are shown in Fig. 8. From the perspective of practical detection, here we give the temperature and pressure distribution of the surface. Although the cloaking effect is not perfect, compared with the case without a cloak [for example, see Figs. 2(a3) and 2(b3)], we can see the bending of isotherms and isobars in the background region is alleviated to a certain extent. The parameters (including the volume fraction, geometry and thermal properties of the pillars, and the depth of the cloak layer) can be further optimized through analytical techniques and numerical methods.
5. Convergence analysis of simulations

As we adopt the finite element method to model thermal convection, a convergence analysis is necessary for reliability. With refinement of meshes, we can get more accurate and convincing calculation results. As an example, we use five different sets of meshes, numbered 1 to 5, to execute independent simulations for the designed thermal cloak in Fig. 2. The size parameters of each set are listed in Table I. Three groups of data at the positions \((x = \pm 5 \times 10^{-4} \text{ m})\) and \(x = 0\) are extracted to compare the

![Fig. 9](image-url)

**FIG. 9.** Simulation results of the convective cloak using different meshes: temperature and pressure data on (a1),(a2) \(x = -5 \times 10^{-4} \text{ m}\), (b1),(b2) \(x = 0\), and (c1),(c2) \(x = 5 \times 10^{-4} \text{ m}\).
results produced by different grids. It is noted that bigger mesh numbers correspond to more elements, and “Mesh 4” is the actual mesh used in Fig. 2.

We plot the temperature and pressure data read from \( x = -5 \times 10^{-4} \, \text{m}, \, x = 0, \) and \( x = 5 \times 10^{-4} \, \text{m} \) in Fig. 9. “Mesh 4” is illustrated with solid lines whereas its counterparts using other meshes are drawn in dashed lines with different colors. First, we observe Figs. 9(a1), 9(b1), and 9(c1), demonstrating temperature comparisons. The latter two sets of mesh (numbered 4 and 5) produce smoother data lines than the first three sets (numbered 1, 2, and 3). In addition, the difference between “Mesh 4” and “Mesh 5” is very small so that the plots almost coincide, meaning that the simulation results of the temperature have converged to a proper accuracy. The same conclusions can be obtained for the pressure data in Figs. 9(a2), 9(b2), and 9(c2). Therefore, the results we obtained using “Mesh 4” in the previous simulations are credible.