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Ising game: Nonequilibrium steady states of resource-allocation systems

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HIGHLIGHTS

- We develop a new kind of game to characterize the human system behavior under external fields.
- A convertible relationship between the external field and resource ratio is revealed.
- Heuristic human experiments and theoretical calculations are presented to explain our results.

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ABSTRACT

Resource-allocation systems are ubiquitous in the human society. But how external fields affect the state of such systems remains poorly explored due to the lack of a suitable model. Because the behavior of spins pursuing energy minimization required by physical laws is similar to that of humans chasing payoff maximization studied in game theory, here we combine the Ising model with the market-directed resource-allocation game, yielding an Ising game. Based on the Ising game, we show theoretical, simulative and experimental evidences for a formula, which offers a clear expression of nonequilibrium steady states (NESSs). Interestingly, the formula also reveals a convertible relationship between the external field (exogenous factor) and resource ratio (endogenous factor), and a class of saturation as the external field exceeds certain limits. This work suggests that the Ising game could be a suitable model for studying external-field effects on resource-allocation systems, and it could provide guidance both for seeking more relations between NESSs and equilibrium states and for regulating human systems by choosing NESSs appropriately.

1. Introduction

A typical resource-allocation system involves a large number of intelligent agents. These agents interact with each other and compete for limited resource [1–7]. Such systems are ubiquitous in human society, *e.g.*, competition among companies, selection of driving routes, betting on horse racing, *etc*. Since the emergence of minority game, plenty of game models are developed [8–16]. They pave the way for solving such complicated social problems, although some of them may be oversimplified. However, how external fields (*e.g.*, macroscopic policies or public information) affect the state of such systems remains poorly explored due to the lack of a suitable model. It is known that agents' decisions are usually determined by the previous performances of their strategies as well as by the ever-changing system environment. In this sense, we believe that such systems are analogue to ferromagnetic materials which can be described by Ising model [17–20]. Details of the analogy are shown as follows.

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The agents are like spins, and the environment of the system is similar to an external magnetic field imposed on these spins [21]. An agent's choice among different behaviors is like a spin's occupation of one of its possible states, and the change of agents' behaviors guided by strategies resembles the change of spin states required by physical laws. In particular, the behavior of spins pursuing energy minimization is similar to that of agents seeking payoff maximization in a resource-allocation system. This consistency is natural because it does not rely on specific parameter choices. It is worth noting that the postulate for such analogy lies on the simplification (assumption) that determinants of human strategies can be quantified (into numerical profits) and people make their decisions by choosing the behavior(s) with the maximal profit.

Consequently, inspired by the distinguished minority game [22–26], we propose a new game by combining the Ising model with the market-directed resource-allocation game [10–12] (which is an extended version of minority game). For clarity, we entitle this new game "Ising game", which helps to study the effects of external fields on resource-allocation systems. To this end, we report a formular expression for the nonequilibrium steady states (NESSs) [27,28] of such systems, by performing theoretical analysis, computer simulations, and experimental research. The formula also reveals a convertible relationship between the external field (exogenous factor) and resource ratio (endogenous factor), and indicates a class of nonlinear behavior as the external field reaches certain limits. Below we shall introduce the relevant theory, simulations, and experiments, respectively.

2. Theory

Let us start by considering an Ising model where N spins with orientations of up or down ($\sigma_i = \pm 1, i = 1, \ldots, N$) are settled on a particular lattice configuration. Each spin only interacts with its first neighborhood. Thus, the energy of spin i is

$$-\sum_j J_{ji}\sigma_i\sigma_j - \mu B\sigma_i. \tag{1}$$

Here, J_{ji} is the exchange energy between spin i and its nearest neighborhood j, B is an external magnetic field affecting all the spins simultaneously, and μ is magnetic permeability. Therefore, the Hamiltonian of the system is

$$-\sum_{\langle i,j\rangle} J_{ji}\sigma_i\sigma_j - \sum_i \mu B\sigma_i, \tag{2}$$

where $\langle i, j\rangle$ denotes a pair of nearest interacting spins. The Hamiltonian would become minimal in equilibrium states.

Likewise, in an N-player consecutive resource-allocation game, player i has to make a decision s_i (such as, cooperation or defection in negotiations, taking actions or holding positions in stock markets, *etc.*) in each round. The immediate payoff of player i depends on all the players' choices and can be written as $U_i'(s_1, s_2, \ldots, s_N)$. In an environment providing extra self-related payoff, the total payoff of player i is given by

$$U_i = U_i'(s_1, s_2, \ldots, s_N) + U''(s_i). \tag{3}$$

Economic equilibrium states are reached even though all the players are pursuing self-payoff maximization. In such kind of consecutive games, the system will approach one of the Nash equilibria [29] after sufficient rounds.

In order to study the effect of external fields on resource-allocation systems, we propose an Ising game by combining the Ising model and the market-directed resource-allocation game [10]. In the Ising game, agents can choose between two rooms to enter. Whoever entering a particular room acquires the access of the resource in it. In each round, the room in which agents obtain more average resource becomes the winning side. Specially, a parameter h representing the system environment (the external field) is introduced into the payoffs of agents. The analogy of Ising model and Ising game is illustrated in Fig. 1.

We set the detailed rules for our Ising game as follows: there are two virtual rooms, *i.e.*, Room 1 and Room 2. Either room has a certain amount of resource denoted as M_1 and M_2, respectively. In each round, each player has to decide which room to enter. If it turns out that $M_1/N_1 > M_2/N_2$ (here N_i denotes the number of players entering Room i in a particular round), the players in Room 1 win with a gain of b points individually, and vice versa. In addition, we give extra h points to players entering Room 1, which is pre-known to all. Thus, a positive (negative) value of h indicates a subsidy on Room 1 (Room 2). The complete payoff matrix is shown in Table 1. So the payoff of player i can be written as

$$U_i = b\left(1 - \left\lfloor \frac{\sum_{j\neq i} s_i s_j}{N} - \frac{M_1 - M_2}{M_1 + M_2} + 1 \right\rfloor\right) + h\delta_{s_i,1}, \tag{4}$$

where s_i equals 1 for the choice of Room 1 and 0 for the choice of Room 2 of player i, and δ is the Kronecker delta. The value in the brackets equals 1 for those on the winning side and 0 for the others.

We know that the spins in the Ising model prefer minimal energies and the whole system prefers a minimal Hamiltonian. So in our Ising game system, it is reasonable that each player's payoff reaches a maximal value in steady states, which is consistent with the theory of Nash equilibrium. This means the expected payoff of any player choosing Room 1 (the left side

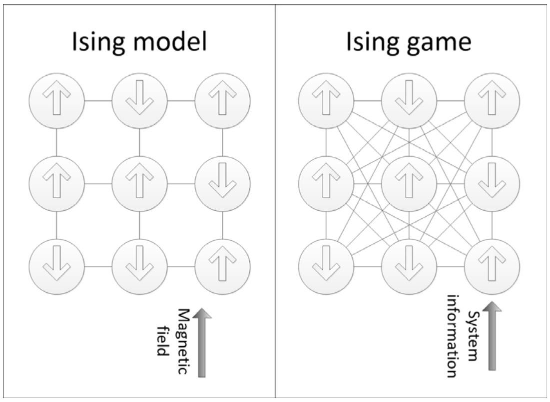

Fig. 1. Schematic graph showing the analogy between the 2-D square lattice Ising model and our Ising game. The up (or down) arrows denote spin-ups (or spin-downs) in the left panel and players' choices to enter Room 1 (or Room 2) in the right panel. The straight solid lines connecting between arrows stand for the interactions between spins or players.

Table 1
Payoff matrix for the players in a single round of Ising game.

Payoff (points)	Win	Lose
Room 1	$b+h$	h
Room 2	b	0

of Eq. (5)) equals the expected payoff of choosing Room 2 (the right side of Eq. (5)). Suppose that there are N players and the probability of each player entering Room 1 is p in the steady state, we have

$$(b+h)\sum_{i=0}^{\left\lfloor (N-1)\frac{m}{m+1}\right\rfloor} C_{N-1}^{i}p^{i}(1-p)^{N-1-i}+h\left[1-\sum_{i=0}^{\left\lfloor (N-1)\frac{m}{m+1}\right\rfloor} C_{N-1}^{i}p^{i}(1-p)^{N-1-i}\right]$$
$$=b\left[1-\sum_{i=0}^{\left\lfloor (N-1)\frac{m}{m+1}\right\rfloor} C_{N-1}^{i}p^{i}(1-p)^{N-1-i}\right], \tag{5}$$

where $m = M_1/M_2$ and C_{N-1}^{i} is the i-combination of a set $N-1$.

To proceed, for convenience, we set

$$\sum_{i=0}^{\left\lfloor (N-1)\frac{m}{m+1}\right\rfloor} C_{N-1}^{i}p^{i}(1-p)^{N-1-i}=p_{1w}. \tag{6}$$

Essentially, p_{1w} denotes the winning rate of Room 1 in a steady state according to the mean-field approximation (we impose a homogeneous external field h on all players and assume all players are rational to maximize their payoffs). Then, Eq. (5) turns to be

$$p_{1w}=\frac{b-h}{2b}\quad \text{as } |h|\le b \tag{7}$$

by renaming p_{1w} using Eq. (6). In addition, in view of real situations, there must exist a nontrivial relation,

$$p_{1w}=0\quad \text{as } h>b,\qquad p_{1w}=1\quad \text{as } h<-b. \tag{8}$$

Eqs. (7) and (8) indicate that, with a given winning reward b, the winning rates of the two rooms are only determined by the intensity of the external field h. Noteworthily, beyond the limits of h, one of the two pure strategies gets dominated; see Fig. 2.

Once we set the values of h, m, and N, we can exactly obtain the value of p. Here we use some extreme cases to validate the theory. Let $m = 1$ and $N \gg 1$. If $h \ge b$, then $p_{1w} = 0$, and thus $p = 1$. This means that, for a sufficiently large external force, all players will behave the same. If $h = 0$, then $p_{1w} = 0.5$ and $p = 0.5$. This just means that all players choose randomly. These two cases correspond to the phenomena of paramagnetism under large and zero magnetic fields, respectively. Furthermore, if N and p are known for a steady state, we can get a relationship between the external field h and the resource allocation ratio m. Note that the resource allocation ratio describes the imbalance of resource within the

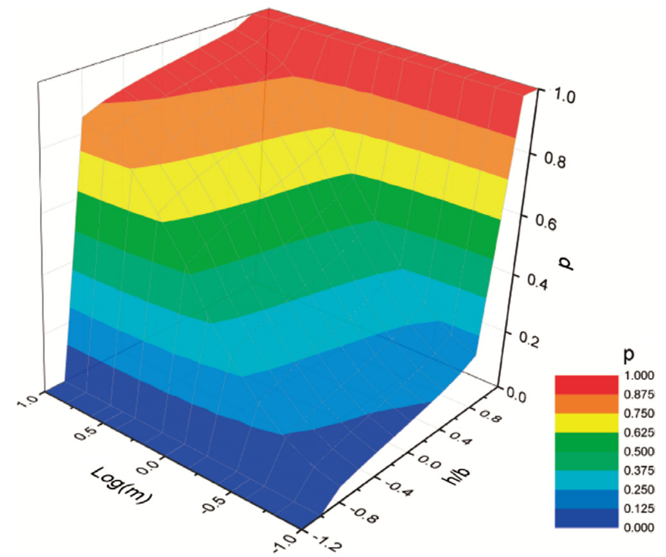

Fig. 2. (Color online) The probability, p, of agents choosing to enter Room 1 in steady states as a function of m (resource allocation ratio, $m = M_1/M_2$) and h/b (h: subsidy given to Room 1; b: basic winning reward).

rooms, while the external field describes an external force imposed on the system. We say h and m are convertible, based on the fact that different combinations of these two parameters can induce the same performance of the system. Fig. 2 shows the probability of agents entering Room 1 under different combinations of m and h. It can be seen that increasing either m or h brings up p. However, the value change of p is much more sensitive to m than to h. The reason is that when we increase h, p_{1w} decreases, which offsets the advantage of the increased payoff. Moreover, the system is saturated for sufficiently large h and m values, which means all agents choose to enter the same room. So the whole system performs as a bi-saturate system, while NESSs are filled between the two saturated states with relatively limited h and m values indicating mixed strategies used by players.

To understand the NESSs, we define an effective entropy R based on the Shannon entropy in information theory to illustrate the degree of uncertainty in our Ising game system [30–32],

$$R = -(f_1 \log_2 f_1 + f_2 \log_2 f_2), \tag{9}$$

where

$$f_1 = \frac{\langle N_1 \rangle / M_1}{\langle N_1 \rangle / M_1 + \langle N_2 \rangle / M_2}, \qquad f_2 = 1 - f_1. \tag{10}$$

Here $\langle N_1 \rangle$ (or $\langle N_2 \rangle$) represents the time average of the number of players entering Room 1 (or Room 2) throughout the whole game process. We find that the Ising game system reaches an equilibrium state in which the value of R equals the maximal value, 1, under zero external field ($h = 0$). However, under a nonzero external field, the system approaches an NESS with $R < 1$ [33–36]. Fig. 3 shows R as a function of h and m (b is set to be 10 points for easy calculations). It further reveals the convertible relationship between the external field (h) and resource allocation ratio (m), since different combinations of h and m values can produce steady states with the same value of R.

3. Simulation and experiment

To validate our theory, we are in a position to perform agent-based simulations according to the market-directed resource-allocation game [10–12]. To emphasize on our study of the influence of the external field, we give an extra subsidy of h points to those agents choosing Room 1, regardless of the final result of each round of game. The detailed mechanism for the model is set as follows. Each agent has s different strategies during the game. Each strategy is a table mapping P historical situations to particular choices of two rooms. In each round, b points are added to a strategy if it has given the correct prediction, and one agent makes a choice based on the prediction of his/her best-scored strategy. The exact meanings of the parameters, the reason for choosing specific parameter values, and detailed mechanisms can be found in [10].

We conducted 6 sessions of computer simulations. Each session contains 10 independent simulations of 1280 rounds with a specific pair of h and m values. We take the last 640 rounds into account while the system is already in a steady state. The results are shown in Fig. 4 and the average performances of the simulations are shown as blue columns in Fig. 7. The behaviors of the time series of the agents' choices and R values are consistent with those of the theory. It can also be seen that the trend of the average number of agents entering Room 1 is close to the theoretical results.

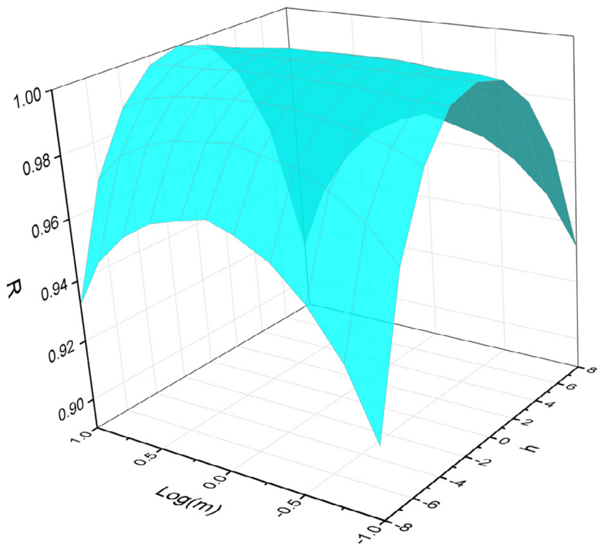

Fig. 3. (Color online) R as a function of h and m. The surface is symmetric with respect to the plain of $h = 0$ or $m = 1$. Clearly, the system reaches equilibrium states ($R = 1$) for $h = 0$, and NESSs ($R < 1$) for $h \neq 0$.

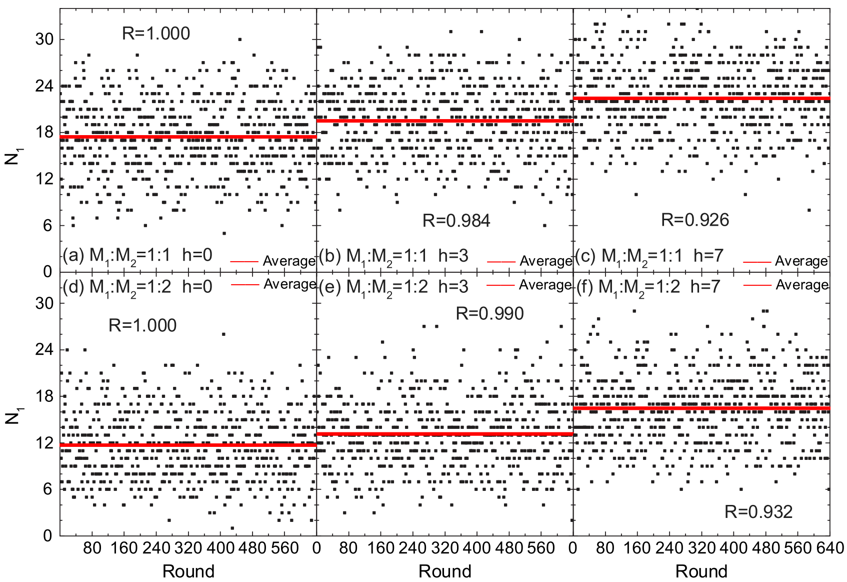

Fig. 4. (Color online) Simulation results: the number of agents entering Room 1 (N_1) in each round with R values displayed for different M_1/M_2 and h. Parameters: $P = 64$, $N = 34$, $s = 32$, and $b = 10$ points.

To further investigate the system properties under different values of h and m, we designed and conducted a series of controlled human experiments. We recruited 34 students from Fudan University to play 6 sessions of Ising game. Each session had 20 rounds which took about 30 minutes. At the beginning of the game, the participants were told the exact rules (*e.g.*, no communications allowed) and payoffs under different circumstances. The value of M_1/M_2 was fixed during a session, but can be distinct among sessions. After each round, the winning room was shown to the participants as well as the payoffs they got. When the whole experiments ended, the total payoff of a participant would be converted to cash (Chinese Yuan) under a fixed exchange rate of 10:1. A photo of the human experiment dated on 20th Sept, 2014 can be found on http://econophysics.fudan.edu.cn/jphuang/.

In the experiments, for $M_1 = M_2$ and $b = 10$ points, we conducted 3 sessions of Ising game. The time series of N_1 and the detailed choice distributions of the participants are shown in Fig. 5. We can see that N_1 fluctuates around a certain value which becomes larger as the value of h grows. This is consistent with the intuition that participants are more likely to choose Room 1 if there is a higher level of subsidy. The R values are 1.000, 0.994, & 0.973 for $h = 0$, 3, & 7, respectively. $R = 1$ for $h = 0$ means that the system with zero external field reaches the equilibrium state. However, for $h \neq 0$, it is obvious that $R < 1$, and the system now approaches an NESS. Fig. 6 shows the experimental results under the condition of $M_1/M_2 = 1/2$.

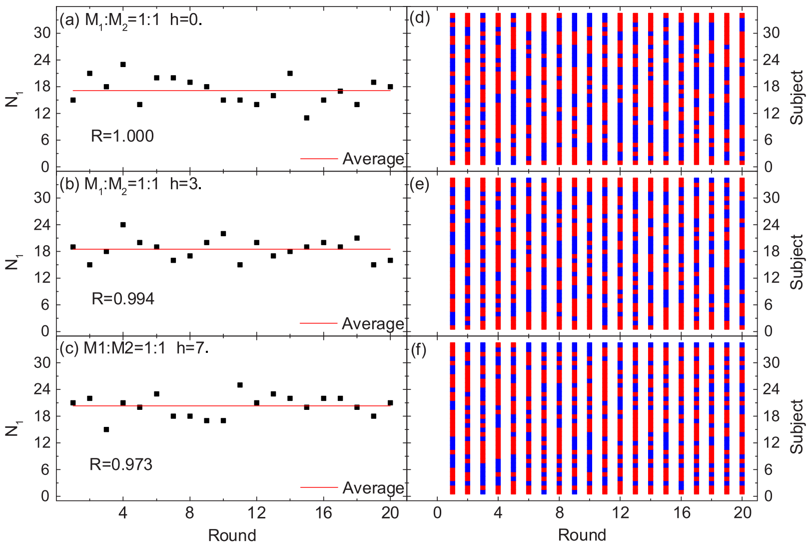

Fig. 5. (Color online) Experimental results: the number of participants entering Room 1 (N_1) in each round with R values displayed for $M_1/M_2 = 1$ and $b = 10$ points. (a) $h = 0$; (b) $h = 3$; (c) $h = 7$; (d)–(f) show the specific choices for each participant in all rounds. The red (blue) patches represent for the choices of Room 1 (Room 2).

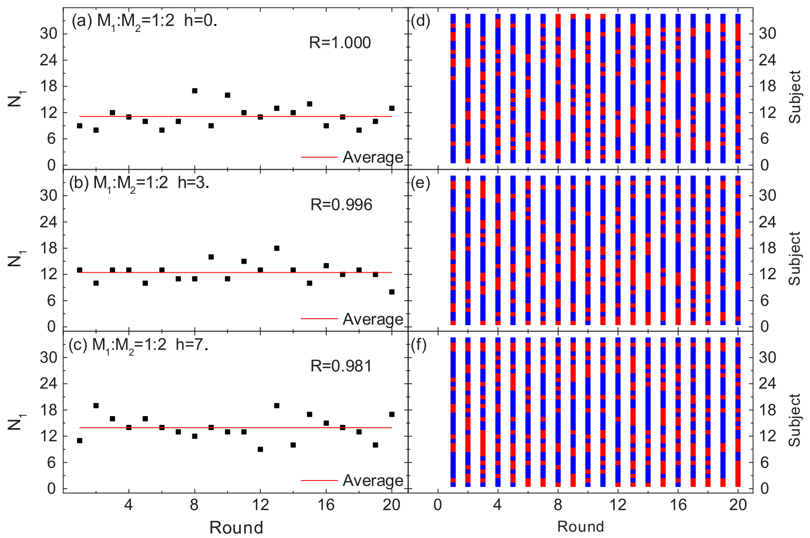

Fig. 6. (Color online) Same as Fig. 5, but for $M_1/M_2 = 1/2$.

Similarly, it can be seen that when h grows from 0, the average value of N_1 becomes larger, and the system evolves away from the equilibrium state to NESSs.

Green columns in Fig. 7 represent the results of our experiments, and Table 2 shows the comparison between theoretical and experimental probabilities of p_{1w} in a session of 20 rounds ($b = 10$ points). Obviously, the experimental results agree with the theory and simulations very well. It is also remarkable that the experimental system can reach the theoretical steady states in only a few rounds. Consequently, here we may also draw a conclusion that the winning rates of the two rooms are only determined by the intensity of the external field.

4. Discussions

The basic intention of our design of Ising game is to study resource-allocation systems under external fields, which former studies did not involve. However, in real world, there are numerous examples of such kinds of fields: government subsidies on production companies, bonuses for workers, information flows in stock markets, *etc*. That is why NESSs should be paid more attention on. Besides, with analogies such as choosing between two rooms and the relative minority wins, there are two significant differences between the Ising game and a large kind of minority games, which make our Ising game a much

Table 2
Probabilities of Room 1 to win in theory and experiments.

$M_1 : M_2$	h	Theoretical	Experimental
1:1	0	0.5	0.475
1:1	3	0.35	0.3
1:1	7	0.15	0.1
1:2	0	0.5	0.55
1:2	3	0.35	0.35
1:2	7	0.15	0.2

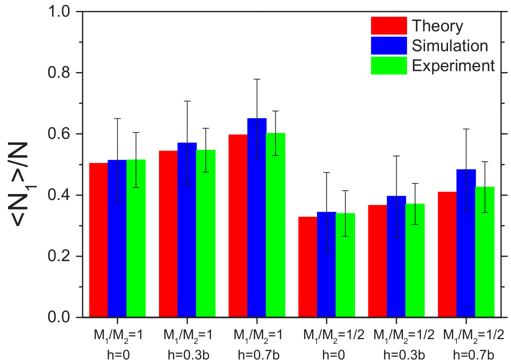

Fig. 7. (Color online) Values of $\langle N_1 \rangle / N$ obtained from theory, simulations, and experiments, with different combinations of M_1/M_2 and h.

precise, general and extended model for analyzing real human-based resource-allocation problems: (1) we do not have exact strategy tables for the agents, while strategy tables are essential in minority game; (2) the nonequilibrium states of the Ising game originate from the induced external field, unlike the frozen agents in minority game who form a spontaneous magnetization.

For administrators and regulators of such a system, our model offers a sound and effective way to tune the system state. It is feasible to dramatically alter the state by adjusting resource ratio m within a limited range. However, if the system is saturated in a strong external field, there is nothing to do with m. By this time, bringing h back to an appropriate scope is the only helpful way.

5. Conclusions

It is known that the research on nonequilibrium statistical physics has been a key point for recent decades. In the duration, theories like the Kubo formula played a significant role in developing nonequilibrium statistical physics of natural systems, but not social systems. Interestingly, the formula [Eq. (5)] obtained in this work starts to offer new hints on nonequilibrium statistical physics of human systems. The formula provides a complete expression of the NESSs of the systems and reveals a convertible relationship between the external field (exogenous factor) and resource ratio (endogenous factor). It also indicates a class of regime changes as external fields vary. Such results echo with reality.

Our Ising game provides a new model to study the behaviors of N-player game systems with external fields, and thus it is also useful for financial markets or game-theory-related fields, such as evolutionism or sociology. For instance, participants in a system might use our method to judge whether the system is in an NESS. On the other hand, regulatory authorities could tune NESSs by adjusting external fields or resource ratios.

To sum up, this work suggests that the Ising game could be a suitable game model for studying external field effects on resource-allocation systems, and it could offer guidance both for seeking more relations between NESSs and equilibrium states of various kinds of systems and for understanding or regulating social human systems by choosing NESSs appropriately.

Acknowledgment

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Appendix. Instructions for the human experiment of Ising game

Basic rules

There will be several sessions of experiments, each containing 20 rounds. In each round of Ising game, the participants should choose Room 1 or Room 2 to enter. There are certain amounts of resources, namely, M_1 and M_2 in the two rooms, respectively, and the participants divide equally the resources of which they have entered. During one particular session of the experiment, M_1 and M_2 will be fixed to a prior-announced value, while they can be different among sessions.

After all the participants have made their choice, the winning side will be determined by those who entered the relatively minority side, i.e., the participants with higher average resource win. Suppose that the numbers of participants entering the two rooms are N_1 and N_2, then those entering Room 1 win if $M_1/N_1 > M_2/N_2$. Otherwise, those entering Room 2 win.

During the whole experiment, no communications are allowed.

Scoring rules

If a participant has made the winning choice, he/she will be awarded 10 points. If he/she loses, nothing is given.

Besides, an extra positive rewarding score of h points will be given to those who chose Room 1. Similar to M_1 and M_2, h is fixed during one session of game, and can be different among sessions. That is, for each participant, there exist four different payoff values under different circumstances:

1. If a participant entered Room 1 and won, he/she would be awarded $10 + h$ points.
2. If a participant entered Room 1 but lost, he/she would be awarded h points.
3. If a participant entered Room 2 and won, he/she would be awarded 10 points.
4. If a participant entered Room 2 but lost, he/she will get nothing.

After each round of game, the game result (i.e., the winning room) and the score they gain are shown to the participants. And everyone's 20 scores of 20 rounds are added up to his/her score of the session. When the whole experiment ends, the total score of a participant will be converted to cash (Chinese Yuan) under a fixed exchange rate, 10:1.

Experiment preparations

1. Go to 10.64.13.200 with web browser Chrome.
2. Register to obtain the access of the experiment. Please use your student ID as the user name. Do not use your normal password! (Simple password is recommended since it is temporarily used.)
3. After registration and login, keep still until the administer announces the beginning of the experiment.

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