Multifrequency cloak with multishell by using transformation medium

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We theoretically investigate multifrequency cloak based on multishell by using transformation medium. As for each shell, we use the Maxwell–Garnett theory and the spectral representation theory to design the permittivity profile for the transverse magnetic wave. The plasma frequencies of the metals used in the inner (B) and outer (C) shells must satisfy $\omega_1 < \omega_{pC} < \omega_2 < \omega_{pB}$. In such case, incident electromagnetic wave of lower frequency ω_1 will be distorted only by the outer shell C without entering the inner shell B to achieve invisibility. However, at the higher operating frequency ω_2, the outer shell C is transparent by choosing the materials appropriately, and the incident electromagnetic wave will be distorted only by the inner shell B to achieve invisibility. So, the multishell cloak can work simultaneously on two different optical frequencies. Furthermore, we use the anisotropic differential effective dipole theory to evaluate the efficiency of the cylindrical cloak. © *2009 American Institute of Physics*. [DOI: 10.1063/1.3147942]

I. INTRODUCTION

There has been considerable interest in the possibility, both theoretical and experimental, of invisibility cloak for the past few years. In 2006, Pendry *et al.*[1] have reported an interesting idea of using the transformation optics approach to design a cloak of invisibility, through which electromagnetic fields can be excluded from a object without perturbing the exterior fields. Independently, an optical conformal mapping method has been used to design a medium that creates perfect invisibility in the geometric limit.[2] The first experimental demonstration of a two-dimensional invisibility cloak at microwave frequency was reported later by using metamaterials.[3] A full-wave computer simulation of the cylindrical cloak with transverse electric (TE) polarization was then performed.[4] And an optical cloaking device with transverse magnetic (TM) polarization was also proposed by using nonmagnetic composite materials.[5] Other methods to achieve invisibility, such as that of Alù and Engheta,[6] proposed a method by using plasmonic and metamaterial to drastically reduce the total scattering cross section of a given object by relying on an inherently nonresonant scattering cancellation phenomenon. The transformation optics method was also extended to other fields, such as acoustic,[7] conductivity,[8] and matter wave,[9] etc. Researchers also considered the cloak for multilayered and gradually changing background media,[10] as well as anisotropic background materials.[8] Due to causality constraints,[11] the perfect cloak can only work at a single fixed frequency. Wang *et al.*[12] proposed a method to use the active metamaterials to extend the operating frequencies. Recently, Leonhardt and Tyc[13] improved the design of invisibility cloaks by showing that curved spaces can be used to hide objects by bending light around them, leading to invisibility in a broad band of the spectrum.

In this work, we theoretically investigate how to extend the operating frequencies based on multishell by using transformation medium. As for each shell, we use the Maxwell–Garnett theory and the spectral representation theory to design the shell to fit the ideal permittivity profile for the TM wave. The plasma frequencies of the metals used in the inner (B) and outer (C) shells must satisfy $\omega_1 < \omega_{pC} < \omega_2 < \omega_{pB}$. In such case, incident electromagnetic wave of lower frequency ω_1 will be distorted only by outer shell C without entering inner shell B to achieve invisibility. However, at higher operating frequency, the outer shell C is transparent by choosing the materials appropriately, and the incident electromagnetic wave will be distorted only by the inner shell B to achieve invisibility. So, the multishell cloak can work simultaneously at two different optical frequencies. In order to evaluate the efficiency of the cloak, we use the anisotropic differential effective dipole theory (ADEDT) to calculate the effective dipole moment of the cylindrical cloak shells.

II. COORDINATE TRANSFORMATION

We consider the cylindrical case, a cylindrical region $r < b$ is compressed into a concentric cylindrical shell $a < r < b$ under the coordinate transformation

$$r' = \frac{b-a}{b} r + a, \quad \theta' = \theta, \quad z' = z. \tag{1}$$

Based on the form invariant of the Maxwell's equations in the original and transformed spaces, we can obtain the permittivity and permeability tensors of the medium in the transformed space[14]

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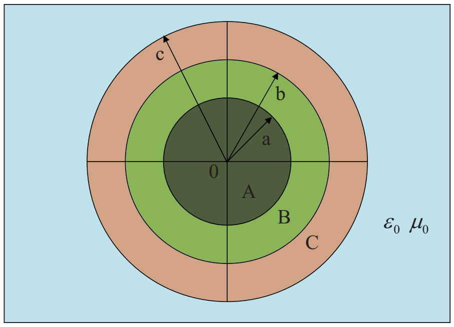

FIG. 1. (Color online) Schematic graph showing is the cross section of the multishell cylindrical cloak. Two shells B with radius b and C with radius c covering core A with radius a.

$$\bar{\varepsilon}' = \frac{\overline{\mathbf{A}} \cdot \overline{\mathbf{A}}^T}{det(\overline{\mathbf{A}})} \cdot \bar{\varepsilon}, \tag{2}$$

$$\bar{\mu}' = \frac{\overline{\mathbf{A}} \cdot \overline{\mathbf{A}}^T}{det(\overline{\mathbf{A}})} \cdot \bar{\mu}, \tag{3}$$

where $\overline{\mathbf{A}}$ is the Jacobian transformation matrix between the transformed coordinate and the original coordinate. So, we easily get the anisotropic permittivity and permeability tensors of the material in the inner cloaking shell B (Ref. 3)

$$\varepsilon_{\mathrm{B}r} = \mu_{\mathrm{B}r} = \frac{r-a}{r}, \tag{4}$$

$$\varepsilon_{\mathrm{B}\theta} = \mu_{\mathrm{B}\theta} = \frac{r}{r-a}, \tag{5}$$

$$\varepsilon_{\mathrm{B}z} = \mu_{\mathrm{B}z} = \left(\frac{b}{b-a}\right)^2 \frac{r-a}{r}. \tag{6}$$

III. DESIGN CLOAKING SHELL BASED ON SPECTRAL REPRESENTATION THEORY

For simplicity, let us consider two operating frequencies ω_1 and ω_2, with $\omega_2 = 2\omega_1$, and two shells B and C covering core A, as shown in Fig. 1. For TM wave, with the magnetic field polarized along the z-axis. only $\varepsilon_{\mathrm{B}r}$, $\varepsilon_{\mathrm{B}\theta}$, and $\mu_{\mathrm{B}z}$ in Eqs. (4)–(6), enter the Maxwell's equations. The phase front and power-flow bending inside the cloak shell remain unaffected as long as the products $\mu_{\mathrm{B}z}\varepsilon_{\mathrm{B}r}$ and $\mu_{\mathrm{B}z}\varepsilon_{\mathrm{B}\theta}$ are kept the same as the original three. So, we can obtain the reduced cloak parameters for the inner cloaking shell B (Ref. 5)

$$\varepsilon_{\mathrm{B}r} = \left(\frac{b}{b-a}\right)^2 \left(\frac{r-a}{r}\right)^2, \tag{7}$$

$$\varepsilon_{\mathrm{B}\theta} = \left(\frac{b}{b-a}\right)^2, \tag{8}$$

$$\mu_{\mathrm{B}z} = 1. \tag{9}$$

The strategy is to sacrifice the perfectly matched impedance at $r=b$ for nonmagnetic consideration. By using the same method, the properties of the material in the outer shell C for TM wave should be

$$\varepsilon_{\mathrm{C}r} = \left(\frac{c}{c-b}\right)^2 \left(\frac{r-b}{r}\right)^2, \tag{10}$$

$$\varepsilon_{\mathrm{C}\theta} = \left(\frac{c}{c-b}\right)^2, \tag{11}$$

$$\mu_{\mathrm{C}z} = 1. \tag{12}$$

In our design, we distribute nanosize metal particles of ε_1 into a homogeneous dielectric medium of permittivity ε_2 in the cloak shells. The effective permittivity of the shell can be calculated by using the Maxwell–Garnett theory,[15,16]

$$\frac{\varepsilon_r - \varepsilon_2}{g_r\varepsilon_r + (1-g_r)\varepsilon_2} = f\frac{\varepsilon_1 - \varepsilon_2}{g_r\varepsilon_1 + (1-g_r)\varepsilon_2}, \tag{13}$$

$$\frac{\varepsilon_t - \varepsilon_2}{g_t\varepsilon_t + (1-g_t)\varepsilon_2} = f\frac{\varepsilon_1 - \varepsilon_2}{g_t\varepsilon_1 + (1-g_t)\varepsilon_2}, \tag{14}$$

where ε_r is the radial effective permittivity, ε_t is the tangential effective permittivity, and f denotes the volume fraction of the metal particles. g_r is the depolarization factor along the radial axis, while g_t is the depolarization factor along the tangential axis. There is a geometrical sum rule for g_r and g_t, $2g_t + g_r = 1$. The spectral representation theory can offer the advantage of separating the materials parameters from the particle structure information. We can adjust the material parameters and geometric parameters independently. So we use the spectral representation theory to simplify the design process and introduce a dimensionless parameter s

$$s = \frac{1}{1 - \dfrac{\varepsilon_1}{\varepsilon_2}}. \tag{15}$$

Now, we can obtain the effective permittivity along the radial axis and the tangential axis, respectively,

$$\varepsilon_r = \varepsilon_2\left[1 - \frac{f}{s - g_r(1-f)}\right], \tag{16}$$

$$\varepsilon_t = \varepsilon_2\left[\frac{1 + f - g_r + fg_r - 2s}{(f-1)(g_r-1) - 2s}\right]. \tag{17}$$

To fit the permittivity profile Eqs. (7) and (8), we get the volume fraction and depolarization factor along the radial direction in the inner shell B

$$f(r) = -\frac{(-1+3s)[b^4(a-r)^2 - (a-b)^2b^2(a^2 - 2ar + 2r^2)\varepsilon_2 + (a-b)^4r^2\varepsilon_2^2]}{b^4(a-r)^2 - 2(a-b)^4r^2\varepsilon_2^2 + (a-b)^2b^2(a-r)^2\varepsilon_2}, \tag{18}$$

$$g_r(r) = \frac{b^4(a-r)^2 s + (a-b)^2 b^2\{a^2 s + r[r-2(a+r)s]\}\varepsilon_2 + (a-b)^4 r^2(-1+s)\varepsilon_2^2}{3b^4(a-r)^2 s - (a-b)^2 b^2[ar(4-6s) + 3r^2(-1+2s) + a^2(-2+3s)]\varepsilon_2 + 3(a-b)^4 r^2(-1+s)\varepsilon_2^2}. \tag{19}$$

As for the outer shell C, to fit the permittivity profile Eqs. (10) and (11), we can obtain the volume fraction and depolarization factor along the radial direction by changing a to b, and b to c in Eqs. (18) and (19), respectively.

In real fabrication, the particle is nonspherical particle. For a prolate spheroid with three principal axes, a_1, b_1, and c_1, there is $a_1=b_1<c_1$, and the depolarization factor g_L along the major axis c_1 is[17,18]

$$g_L = \frac{1}{k^2-1}\left[\frac{k}{\sqrt{k^2-1}}\ln(k+\sqrt{k^2-1})-1\right], \tag{20}$$

where $k=c_1/a_1>1$ and $g_L<1/3$. For an oblate spheroid with $a_2<b_2=c_2$, the depolarization factor g_L along the major axis a_2 is[19]

$$g_L = 1-\left[\frac{k^2}{(k^2-1)^{3/2}}\arcsin\frac{\sqrt{k^2-1}}{k}-\frac{1}{k^2-1}\right], \tag{21}$$

where $k=c_2/a_2>1$ and $g_L>1/3$. For both cases, there is a geometrical sum rule for g_L and g_T, $2g_T+g_L=1$, where g_T denotes the depolarization factor along the minor axis. It is worth noting that the spherical particle corresponds to $g_L=g_T=1/3$. In our design, we divide the cloak shell into many thin shells, the particle shape factor k in each shell can be calculated from Eqs. (19)–(21).

The dielectric constant of the metal particles used in the cloaking shells are described by the Drude model

$$\epsilon(\omega) = 1-\frac{\omega_p^2}{\omega(\omega+i\gamma)}, \tag{22}$$

where ω_p is the plasma frequency and γ is the collision frequency, related to energy loss. Here, we consider the ideal case, which means $\gamma=0$. ω_{pC} is the plasma frequency of the metal used in the outer cloaking shell C, while ω_{pB} is the plasma frequency of the metal used in the inner cloaking shell B. ω_1 is the operating frequency of the outer cloaking shell C, while ω_2 is the higher operating frequency of the inner cloaking shell B. The plasma frequencies of the metals used in the cloaking shells must satisfy $\omega_1<\omega_{pC}<\omega_2<\omega_{pB}$. In such case, incident electromagnetic wave of lower frequency ω_1 will be distorted only by outer shell C without entering inner shell B to achieve invisibility. However, incident electromagnetic wave of higher frequency ω_2 will treat shell C as transparent and will be distorted only by the inner shell B to achieve invisibility. Thus the outer shell C must be transparent $(n=1)$ to the higher operating frequency ω_2, at the same time, the material in the outer shell C satisfies the cloaking permittivity profile at the lower operating frequency ω_1. In this way, anything inside core A must be invisible at the two different operating frequencies. At the higher operating frequency ω_2, the out shell C should be transparent. To meet the requirement, the host material of shell C is also dispersive. At the higher operating frequency ω_2, the geometrical parameters of the outer shell c has already been fixed to fit the requirements of the operating frequency ω_1. However, we can adjust the properties of the material used in the outer shell C. In our design, $\epsilon_2(\omega_2)$ is slightly larger than unity, and $\epsilon_1(\omega_2)$ is smaller than unity, leading to the effective permittivity of the outer shell C being about unity at the higher operating frequency ω_2. In such case, the outer shell C is almost transparent to the electromagnetic field with higher frequency. So, the multishell cloak can work simultaneously at two different optical frequencies.

IV. NUMERICAL RESULTS

We are now in a position to investigate further the geometric parameters and material parameters of the cloaking shells to achieve invisibility. As for the inner shell B, we choose $s=0.35$, $\omega_2=0.28\omega_{pB}$, and $\varepsilon_2=6.5$. Figure 2 displays the volume fraction of the metal particles f and the depolarization factor along the radial direction g_r as a function of r for different inner radii a. We found that the volume fraction with values smaller than 1, and changes a little as the radius increases. As the inner radius increases, the volume fraction decreases. Also, the depolarization factor are larger than 1/3. As the inner radius increases, the depolarization factor increases. While for the outer shell C, we choose $s=0.7$, $\omega_1=0.67\omega_{pC}$, and $\epsilon_2(\omega_1)=3.0$. Figure 3 shows the volume fraction of the metal particles f and the depolarization factor

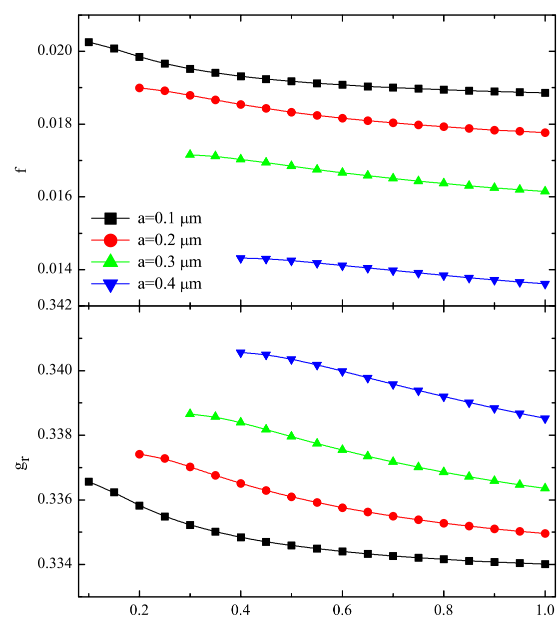

FIG. 2. (Color online) For the inner shell B, the volume fraction of the metal particles f and the depolarization factor along the radial direction g_r as a function of r for different inner radii a. Parameters: $s=0.35$, $\omega_2=0.28\omega_{pB}$, and $\varepsilon_2=6.5$.

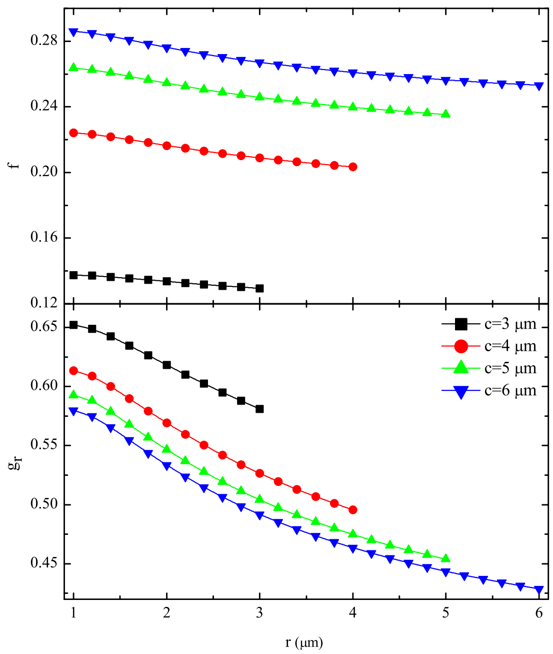

FIG. 3. (Color online) For the outer shell C, the volume fraction of the metal particles f and the depolarization factor along the radial direction g_r as a function of r for different outer radii c. Parameters: $s=0.7$, $\omega_1=0.67\omega_{pC}$, and $\epsilon_2(\omega_1)=3.0$.

along the radial direction g_r as a function of r for different outer radii c. The results are opposite to the inner shell B. As shown in Fig. 4, the effective permittivities ε_r and ε_t of the outer shell C are approaching to unity. Also, the outer shell C is almost isotropic. So the outer shell C is almost transparent for the electromagnetic field with higher frequency.

We can use a numerical method, namely, ADEDT to evaluate the efficiency of the cloak. The ADEDT is indeed exact for anisotropic graded cylindrical particles with arbitrary gradation permittivity profile, as compared to a first-principles approach in Ref. 20. The differential equation about the effective dipole factor can be written as[20]

$$\frac{db(r)}{dr} = \frac{1+\delta}{4r\delta\varepsilon_{rr}(r)\varepsilon_h}\{\varepsilon_{rr}(r)\varepsilon_{\varphi\varphi}(r)[1-b(r)]^2 - \varepsilon_h^2[1 + b(r)]^2\}, \tag{23}$$

where $\delta=\sqrt{\varepsilon_{\varphi\varphi}/\varepsilon_{rr}}$. We can calculate the effective dipole moment by using the ADEDT. Equation (23) can be solved numerically by using the fourth-order Runge–Kutta algorithm. As shown in Figs. 5 and 6, the vanishing effective dipole moment at the outer radii in the cloak shells demonstrates that the cloak is almost perfect.

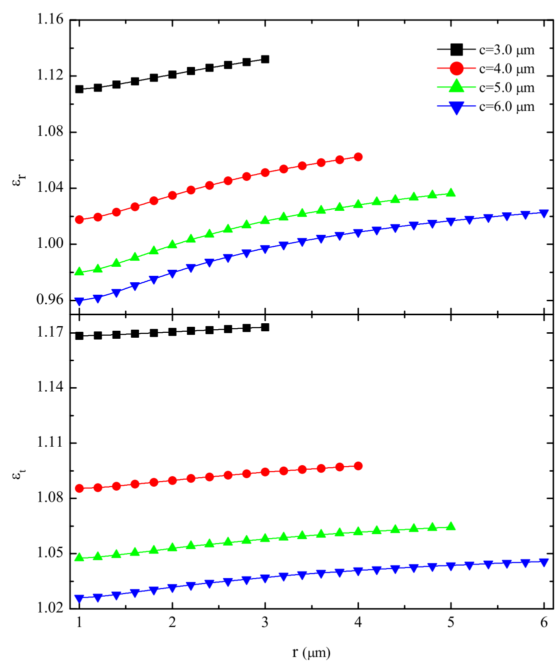

FIG. 4. (Color online) The effective permittivities ε_r and ε_t of the outer shell C as a function of r for different outer radii c at the higher operating frequency. Parameters: $\omega_2=1.33\omega_{pC}$, and $\varepsilon_2(\omega_2)=1.3$.

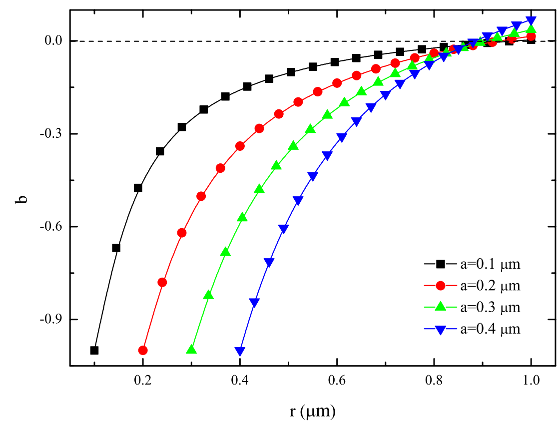

FIG. 5. (Color online) For the inner shell B, the effective dipole moment as a function of r for different inner radii a. Parameters: $s=0.35$, $\omega_2=0.28\omega_{pB}$, and $\varepsilon_2=6.5$.

V. CONCLUSIONS AND DISCUSSION

To sum up, we theoretically investigate multifrequency cloak based on multishell by using transformation optics method. For each cloaking shell, we utilize the Maxwell–Garnett theory and the spectral representation theory to design the shell to fit the perfect permittivity profile for the TM wave. The design process becomes simpler by invoking the spectral representation theory because the spectral representation theory can offer the advantage of separating the mate-

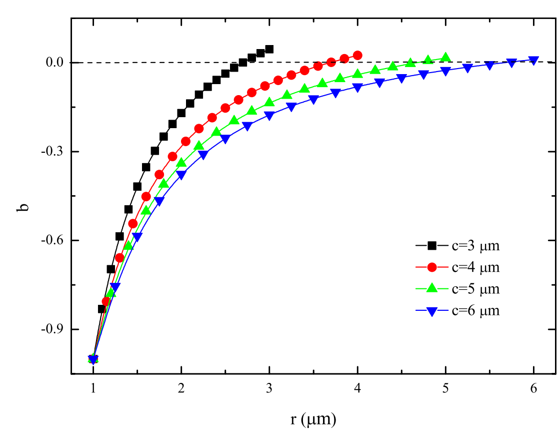

FIG. 6. (Color online) For the outer shell C, the effective dipole moment as a function of r for different outer radii c. Parameters: $s=0.7$, $\omega_1=0.67\omega_{pC}$, and $\epsilon_2(\omega_1)=3.0$.

rial parameters from the particle structure information. So, we can adjust the material parameters and geometric parameters independently in fabrication. The relation between the plasma frequencies of metals used in the cloaking shells and the operating frequencies should be $\omega_1 < \omega_{pC} < \omega_2 < \omega_{pB}$. In such case, incident electromagnetic wave of lower frequency ω_1 will be distorted only by outer shell C without entering inner shell B to achieve invisibility. However, at the higher operating frequency, the outer shell C is transparent by choosing the materials appropriately and the electromagnetic wave will be distorted only by the inner B shell to achieve invisibility. The multishell cloak can work on two different optical frequencies simultaneously. In fact, we can achieve invisibility simultaneously at many frequencies by using the similar design. In order to evaluate the efficiency of the cloak, we use the ADEDT to calculate the effective dipole moment of the cloak shells. The vanishing dipole moment at the outer radii in the cloak shells demonstrate that the cloak is almost perfect. Here, we consider the ideal case, in real case, we should pay much attention to the energy loss in metal. In other words, it is very hard to make the perfect cloak without reflection, shadow, and absorption in real case.

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