9
Bessel Functions

9.1 Bessel Functions of the First Kind
Friedrich Bessel (1784–1846) invented functions for problems with circular
symmetry. The most useful ones are defined for any integer \( n \) by the series

\[
J_n(z) = \frac{z^n}{2^n n!} \left[ 1 - \frac{z^2}{2(2n+2)} + \frac{z^4}{2 \cdot 4(2n+2)(2n+4)} - \ldots \right]
= \left( \frac{z}{2} \right)^n \sum_{m=0}^{\infty} \frac{(-1)^m}{m! (m + n)!} \left( \frac{z}{2} \right)^{2m}.
\]

(9.1)

The first term of this series says that \( J_n(z) \approx z^n / 2^n n! \) for small \( |z| \ll 1 \). The
alternating signs in (9.1) make the waves plotted in Fig. 9.1, and we have
for big \( |z| \gg 1 \) the approximation (Courant and Hilbert, 1955, chap. VII)

\[
J_n(z) \approx \sqrt{\frac{2}{\pi z}} \cos \left( z - \frac{n\pi}{2} - \frac{\pi}{4} \right) + O(|z|^{-3/2}).
\]

(9.2)

The \( J_n(z) \) are entire transcendental functions. They obey Bessel’s equation

\[
\frac{d^2J_n}{dz^2} + \frac{1}{z} \frac{dJ_n}{dz} + \left( 1 - \frac{n^2}{z^2} \right) J_n = 0
\]

(6.315) as one may show (exercise 9.1) by substituting the series (9.1) into
the differential equation (9.3). Their generating function is

\[
\exp \left[ \frac{z}{2} \left( u - 1/u \right) \right] = \sum_{n=-\infty}^{\infty} u^n J_n(z)
\]

(9.4)
The Bessel Functions $J_0(\rho)$, $J_1(\rho)$, and $J_2(\rho)$

Figure 9.1 Top: Plots of $J_0(\rho)$ (solid curve), $J_1(\rho)$ (dot-dash), and $J_2(\rho)$ (dashed) for real $\rho$. Bottom: Plots of $J_3(\rho)$ (solid curve), $J_4(\rho)$ (dot-dash), and $J_5(\rho)$ (dashed). The points at which Bessel functions cross the $\rho$-axis are called zeros or roots; we use them to satisfy boundary conditions.

from which one may derive (exercise 9.5) the series expansion (9.1) and the integral representation (5.46, exercise 9.6)

$$J_n(z) = \frac{1}{\pi} \int_0^\pi \cos(z \sin \theta - n\theta) \, d\theta = J_{-n}(-z) = (-1)^n J_{-n}(z) \quad \text{(9.5)}$$

for all complex $z$. For $n = 0$, this integral is (exercise 9.7) more simply

$$J_0(z) = \frac{1}{2\pi} \int_0^{2\pi} e^{iz \cos \theta} \, d\theta = \frac{1}{2\pi} \int_0^{2\pi} e^{iz \sin \theta} \, d\theta. \quad \text{(9.6)}$$

These integrals (exercise 9.8) give $J_n(0) = 0$ for $n \neq 0$, and $J_0(0) = 1$.

By differentiating the generating function (9.4) with respect to $u$ and
identifying the coefficients of powers of $u$, one finds the recursion relation

$$J_{n-1}(z) + J_{n+1}(z) = \frac{2n}{z} J_n(z). \tag{9.7}$$

Similar reasoning after taking the $z$ derivative gives (exercise 9.10)

$$J_{n-1}(z) - J_{n+1}(z) = 2 J'_n(z). \tag{9.8}$$

By using the gamma function (section 5.12), one may extend Bessel’s equation (9.3) and its solutions $J_n(z)$ to non-integral values of $n$

$$J_{\nu}(z) = \left(\frac{z}{2}\right)^\nu \sum_{m=0}^{\infty} \frac{(-1)^m}{m!\Gamma(m+\nu+1)} \left(\frac{z}{2}\right)^{2m}. \tag{9.9}$$

Letting $z = ax$ in (9.3), we arrive (exercise 9.11) at the self-adjoint form (6.314) of Bessel’s equation

$$-\frac{d}{dx} \left(x \frac{d}{dx} J_n(ax)\right) + \frac{n^2}{x} J_n(ax) = a^2 x J_n(ax). \tag{9.10}$$

In the notation of equation (6.294), $p(x) = x$, $a^2$ is an eigenvalue, and $\rho(x) = x$ is a weight function. To have a self-adjoint system (section 6.28) on an interval $[0,b]$, we need the boundary condition (6.254)

$$0 = [\rho(J_n v' - J'_n v)]_0^b = [x(J_n v' - J'_n v)]_0^b \tag{9.11}$$

for all functions $v(x)$ in the domain $D$ of the system. Since $p(x) = x$, $J_0(0) = 1$, and $J_n(0) = 0$ for integers $n > 0$, the terms in this boundary condition vanish at $x = 0$ as long as the domain consists of functions $v(x)$ that are continuous on the interval $[0,b]$. To make these terms vanish at $x = b$, we require that $J_n(ab) = 0$ and that $v(b) = 0$. So $ab$ must be a zero $z_{n,m}$ of $J_n(z)$, that is $J_n(ab) = J_n(z_{n,m}) = 0$. With $a = z_{n,m}/b$, Bessel’s equation (9.10) is

$$-\frac{d}{dx} \left(x \frac{d}{dx} J_n \left(\frac{z_{n,m}x}{b}\right)\right) + \frac{n^2}{x} J_n \left(\frac{z_{n,m}x}{b}\right) = \frac{z_{n,m}^2}{b^2} x J_n \left(\frac{z_{n,m}x}{b}\right). \tag{9.12}$$

For fixed $n$, the eigenvalue $a^2 = \frac{z_{n,m}^2}{b^2}$ is different for each positive integer $m$. Moreover as $m \to \infty$, the zeros $z_{n,m}$ of $J_n(x)$ rise as $m\pi$ as one might expect since the leading term of the asymptotic form (9.2) of $J_n(x)$ is proportional to $\cos(x - m\pi + (n+1)/2)$ which has zeros at $m\pi + (n+1)/2 + \pi/4$. It follows that the eigenvalues $a^2 \approx (m\pi)^2/b^2$ increase without limit as $m \to \infty$ in accordance with the general result of section 6.34. It follows then from the argument of section 6.35 and from the orthogonality relation (6.333) that for every fixed $n$, the infinite sequence of eigenfunctions $J_n(z_{n,m}x/b)$,
one for each zero, are complete in the mean, orthogonal, and normalizable
on the interval \([0, b]\) with weight function \(\rho(x) = x\)

\[
\int_0^b x J_n \left( \frac{z_{n,m} x}{b} \right) J_n \left( \frac{z_{n,m} x}{b} \right) \, dx = \delta_{m,m'} \frac{\beta^2}{2} J_n^2(z_{n,m}) = \delta_{m,m'} \frac{\beta^2}{2} J_{n+1}^2(z_{n,m})
\]

(9.13)

and a normalization constant (exercise 9.12) that depends upon the first
derivative of the Bessel function or the square of the next Bessel function at
the zero.

The analogous relation on an infinite interval is

\[
\int_0^\infty x J_n(kx) J_n(k'x) \, dx = \frac{1}{k'} \delta(k - k').
\]

(9.14)

One may generalize these relations (9.10–9.14) from integral \(n\) to real non-negative \(\nu\) (and to \(\nu > -1/2\)).

**Example 9.1 (Bessel’s Drum)** The top of a drum is a circular membrane
with a fixed circumference. The membrane’s potential energy is approxi-
mately proportional to the extra area it has when it’s not flat. Let \(h(x, y)\)
be the displacement of the membrane in the \(z\) direction normal to the \(x-y\)
plane of the flat membrane, and let \(h_x\) and \(h_y\) denote its partial derivatives
(6.20). The extra length of a line segment \(dx\) on the stretched membrane is
\(\sqrt{1 + h_x^2 + h_y^2} \, dx\), and so the extra area of an element \(dx \, dy\) is

\[
dA \approx \left( \sqrt{1 + h_x^2 + h_y^2} - 1 \right) dx \, dy \approx \frac{1}{2} (h_x^2 + h_y^2) \, dx \, dy.
\]

(9.15)

The (non-relativistic) kinetic energy of the area element is proportional to
the square of its speed. So if \(\sigma\) is the surface tension and \(\mu\) the surface mass
density of the membrane, then to lowest order in \(d\) the action functional is

\[
S[h] = \int \left[ \frac{\mu}{2} h_t^2 - \frac{\sigma}{2} (h_x^2 + h_y^2) \right] \, dx \, dy \, dt.
\]

(9.16)

We minimize this action for \(h\)’s that vanish on the boundary \(x^2 + y^2 = r^2\)

\[
0 = \delta S[h] = \int \left[ \mu h_t \delta h_t - \sigma (h_x \delta h_x + h_y \delta h_y) \right] \, dx \, dy \, dt.
\]

(9.17)

Since (6.173) \(\delta h_t = (\delta h)_t, \delta h_x = (\delta h)_x,\) and \(\delta h_y = (\delta h)_y,\) we can integrate
by parts and get

\[
0 = \delta S[h] = \int \left[ - \mu h_t + \sigma (h_{xx} + h_{yy}) \right] \delta h \, dx \, dy \, dt
\]

(9.18)

apart from a surface term proportional to \(\delta h\) which vanishes because \(\delta h = 0\)
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on the circumference of the membrane. The membrane therefore obeys the wave equation

\[ \mu \frac{h_{tt}}{h} = \sigma (h_{xx} + h_{yy}) \equiv \sigma \Delta h. \]  

(9.19)

This equation is separable, and so letting \( h(x, y, t) = s(t) v(x, y) \), we have

\[ \frac{s''}{s} = \frac{\sigma}{\mu} \frac{\Delta v}{v} = -\omega^2. \]  

(9.20)

The eigenvalues of the Helmholtz equation \(- \Delta v = \lambda v\) give the angular frequencies as \( \omega = \sqrt{\sigma \lambda / \mu} \). The time dependence then is

\[ s(t) = a \sin(\sqrt{\sigma \lambda / \mu}(t - t_0)) \]  

(9.21)

in which \( a \) and \( t_0 \) are constants.

In polar coordinates, Helmholtz’s equation is separable (6.45–6.48)

\[ - \Delta v = - v_{rr} - r^{-1} v_r - r^{-2} v_{\theta \theta} = \lambda v. \]  

(9.22)

We set \( v(r, \theta) = u(r) h(\theta) \) and find \(- v''h - r^{-1}u' h - r^{-2}uh'' = \lambda uh\). After multiplying both sides by \( r^2/uh \), we get

\[ r^2 \frac{u''}{u} + r \frac{u'}{u} + \lambda r^2 = - \frac{h''}{h} = n^2. \]  

(9.23)

The general solution for \( h \) then is \( h(\theta) = b \sin(n(\theta - \theta_0)) \) in which \( b \) and \( \theta_0 \) are constants and \( n \) must be an integer so that \( h \) is single valued on the circumference of the membrane.

The function \( u \) thus is an eigenfunction of the self-adjoint differential equation (6.314) \(- (r u')' + n^2 u/r = \lambda r u\) whose eigenvalues \( \lambda \equiv z^2/r_d^2 \) are all positive. By changing variables to \( \rho = z r/r_d \) and letting \( u(r) = J_n(\rho) \), we arrive (exercise 6.22) at

\[ \frac{d^2 J_n}{d\rho^2} + \frac{1}{\rho} \frac{dJ_n}{d\rho} + \left( 1 - \frac{n^2}{\rho^2} \right) J_n = 0 \]  

(9.24)

which is Bessel’s equation (6.315).

The eigenvalues \( \lambda = z^2/r_d^2 \) are determined by the boundary condition \( u(r_d) = J_n(z) = 0 \). For each integer \( n \geq 0 \), there are an infinite number of zeroes \( z_{n,m} \) at which the Bessel function vanishes, \( J_n(z_{n,m}) = 0 \). Thus \( \lambda = \lambda_{n,m} = z_{n,m}^2/r_d^2 \) and so the frequency is \( \omega = (z_{n,m}/r_d) \sqrt{\sigma/\mu} \). The
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general solution to the wave equation (9.19) of the membrane then is

\[
h(r, \theta, t) = \sum_{n=0}^{\infty} \sum_{m=1}^{\infty} c_{n,m} \sin \left( \frac{z_{n,m}}{r_0} \sqrt{\frac{\sigma}{\mu}} (t - t_0) \right) \sin \left[ n(\theta - \theta_0) \right] J_n \left( \frac{z_{n,m}}{r_0} \right)
\]

in which \( t_0 \) and \( \theta_0 \) can depend upon \( n \) and \( m \). For any \( n \), the zeros \( z_{n,m} \) for big \( m \) are near \( m\pi + (n + 1)\pi/2 + \pi/4 \). \( \square \)

We learned in section 6.5 that in three dimensions Helmholtz’s equation

\[- \Delta V = \alpha^2 V \]

separates in cylindrical coordinates (and in several other coordinate systems). That is, the function \( V(\rho, \phi, z) = B(\rho)\Phi(\phi)Z(z) \) satisfies the equation

\[- \Delta V = - \frac{1}{\rho} \left[ (\rho V_{,\rho})_{,\rho} + \frac{1}{\rho} V_{,\phi\phi} + \rho V_{,zz} \right] = \alpha^2 V \quad (9.26)\]

if \( B(\rho) \) obeys Bessel’s equation

\[
\rho \frac{d}{d\rho} \left( \frac{dB}{d\rho} \right) + \left( (\alpha^2 + k^2)\rho^2 - n^2 \right) B = 0 \quad (9.27)
\]

and \( \Phi \) and \( Z \) respectively satisfy

\[
- \frac{d^2 \Phi}{d\phi^2} = n^2 \Phi(\phi) \quad \text{and} \quad \frac{d^2 Z}{dz^2} = k^2 Z(z) \quad (9.28)
\]

or if \( B(\rho) \) obeys the Bessel equation

\[
\rho \frac{d}{d\rho} \left( \frac{dB}{d\rho} \right) + \left( (\alpha^2 - k^2)\rho^2 - n^2 \right) B = 0 \quad (9.29)
\]

and \( \Phi \) and \( Z \) satisfy

\[
- \frac{d^2 \Phi}{d\phi^2} = n^2 \Phi(\phi) \quad \text{and} \quad \frac{d^2 Z}{dz^2} = -k^2 Z(z). \quad (9.30)
\]

In the first case (9.36 & 9.37), the solution \( V \) is

\[
V_{k,n}(\rho, \phi, z) = J_n(\sqrt{\alpha^2 + k^2}\,\rho)e^{\pm in\phi}e^{\pm kz} \quad (9.31)
\]

while in the second case (9.36 & 9.37) it is

\[
V_{k,n}(\rho, \phi, z) = J_n(\sqrt{\alpha^2 - k^2}\,\rho)e^{\pm in\phi}e^{\pm ikz}. \quad (9.32)
\]

In both cases, \( n \) must be an integer if the solution is to be single valued on the full range of \( \phi \) from 0 to \( 2\pi \).
When \( \alpha = 0 \), the Helmholtz equation reduces to the Laplace equation \( \nabla^2 V = 0 \) of electrostatics which the simpler functions

\[
V_{k,n}(\rho, \phi, z) = J_n(k\rho)e^{\pm in\phi}e^{\pm kz} \quad \text{and} \quad V_{k,n}(\rho, \phi, z) = J_n(ik\rho)e^{\pm in\phi}e^{\pm kz}
\]

satisfy. The product \( i^{-\nu} J_\nu(ik\rho) \) is real and is known as the modified Bessel function

\[
I_\nu(ik\rho) \equiv i^{-\nu} J_\nu(ik\rho).
\]

Modified Bessel functions occur in solutions of the diffusion equation \( \Delta V = \alpha^2 V \). The function \( V(\rho, \phi, z) = B(\rho)\Phi(\phi)Z(z) \) satisfies

\[
\nabla^2 V = \frac{1}{\rho} \left[ (\rho V_\rho)_\rho + \frac{1}{\rho} V_{\phi\phi} + \rho V_{zz} \right] = \alpha^2 V
\]

if \( B(\rho) \) obeys Bessel’s equation

\[
\rho \frac{d}{d\rho} \left( \frac{dB}{d\rho} \right) - \left( (\alpha^2 - k^2)\rho^2 + n^2 \right) B = 0
\]

and \( \Phi \) and \( Z \) respectively satisfy

\[
- \frac{d^2\Phi}{d\phi^2} = n^2\Phi(\phi) \quad \text{and} \quad \frac{d^2Z}{dz^2} = k^2Z(z)
\]

or if \( B(\rho) \) obeys the Bessel equation

\[
\rho \frac{d}{d\rho} \left( \rho \frac{dB}{d\rho} \right) - \left( (\alpha^2 + k^2)\rho^2 + n^2 \right) B = 0
\]

and \( \Phi \) and \( Z \) satisfy

\[
- \frac{d^2\Phi}{d\phi^2} = n^2\Phi(\phi) \quad \text{and} \quad \frac{d^2Z}{dz^2} = -k^2Z(z).
\]

In the first case (9.36 & 9.37), the solution \( V \) is

\[
V_{k,n}(\rho, \phi, z) = I_n(\sqrt{\alpha^2 - k^2}\rho)e^{\pm in\phi}e^{\pm kz}
\]

while in the second case (9.36 & 9.37) it is

\[
V_{k,n}(\rho, \phi, z) = I_n(\sqrt{\alpha^2 + k^2}\rho)e^{\pm in\phi}e^{\pm ikz}.
\]

In both cases, \( n \) must be an integer if the solution is to be single valued on the full range of \( \phi \) from 0 to \( 2\pi \).

**Example 9.2** (Charge near a Membrane) We will use \( \rho \) to denote the density of free charges—those that are free to move in or out of a dielectric medium, as opposed to those that are part of the medium, bound in it by
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molecular forces. The time-independent Maxwell equations are Gauss’s law \( \nabla \cdot \mathbf{D} = \rho \) for the divergence of the electric displacement \( \mathbf{D} \), and the static form \( \nabla \times \mathbf{E} = 0 \) of Faraday’s law which implies that the electric field \( \mathbf{E} \) is the gradient of an electrostatic potential \( \mathbf{E} = -\nabla V \).

Across an interface between two dielectrics with normal vector \( \hat{n} \), the tangential electric field is continuous, \( \hat{n} \times \mathbf{E}_2 = \hat{n} \times \mathbf{E}_1 \), while the normal component of the electric displacement jumps by the surface density \( \sigma \) of free charge, \( \hat{n} \cdot (\mathbf{D}_2 - \mathbf{D}_1) = \sigma \). In a linear dielectric, the electric displacement \( \mathbf{D} \) is the electric field multiplied by the permittivity \( \epsilon \) of the material, \( \mathbf{D} = \epsilon \mathbf{E} \).

The membrane of a eukaryotic cell is a phospholipid bilayer whose area is some \( 3 \times 10^8 \) nm\(^2\), and whose thickness \( t \) is about 5 nm. On a scale of nanometers, the membrane is flat. We will take it to be a slab extending to infinity in the \( x \) and \( y \) directions. If the interface between the lipid bilayer and the extracellular salty water is at \( z = 0 \), then the cytosol extends thousands of nm down from \( z = -t = -5 \) nm. We will ignore the phosphate head groups and set the permittivity \( \epsilon_{\ell} \approx 2\epsilon_0 \); the permittivity of the extracellular water and that of the cytosol are \( \epsilon_w \approx \epsilon_c \approx 80\epsilon_0 \).

We will compute the electrostatic potential \( V \) due to a charge \( q \) at a point \( (0,0,\rho) \) on the \( z \)-axis above the membrane. This potential is cylindrically symmetric about the \( z \)-axis, so \( V = V(q, z) \). The functions \( J_0(k\rho) e^{ikz} \) form a complete set of solutions of Laplace’s equation, but due to the symmetry, we only need the \( n = 0 \) functions \( J_0(k\rho) e^\pm k\rho \). Since there are no free charges in the lipid bilayer or in the cytosol, we may express the potential in the lipid bilayer \( V_{\ell} \) and in the cytosol \( V_c \) as

\[
V_{\ell}(\rho, z) = \int_0^\infty dk J_0(k\rho) \left[ m(k) e^{kz} + f(k) e^{-kz} \right],
\]

\[
V_c(\rho, z) = \int_0^\infty dk J_0(k\rho) d(k) e^{kz}.
\]

The Green’s function (3.110) for Poisson’s equation \(-\nabla^2 G(\mathbf{x}) = \delta^{(0)}(\mathbf{x})\) in cylindrical coordinates is (5.139)

\[
G(\mathbf{x}) = \frac{1}{4\pi|\mathbf{x}|} = \frac{1}{4\pi \sqrt{\rho^2 + z^2}} = \int_0^\infty \frac{dk}{4\pi} J_0(k\rho) e^{-k|z|}.
\]

Thus we may expand the potential in the salty water as

\[
V_w(\rho, z) = \int_0^\infty dk J_0(k\rho) \left[ \frac{q}{4\pi \epsilon_w} e^{-k|z-\rho|} + u(k) e^{-kz} \right].
\]

Using \( \hat{n} \times \mathbf{E}_2 = \hat{n} \times \mathbf{E}_1 \) and \( \hat{n} \cdot (\mathbf{D}_2 - \mathbf{D}_1) = \sigma \), suppressing \( k \), and setting
\[ \beta \equiv q e^{-kh}/4\pi \epsilon_0 \epsilon_w \text{ and } y = e^{2kt} \], we get four equations
\[
m + f - u = \beta \quad \text{and} \quad \epsilon_\ell m - \epsilon_\ell f + \epsilon_w u = \epsilon_w \beta \]
\[
\epsilon_\ell m - \epsilon_\ell y f - \epsilon_c d = 0 \quad \text{and} \quad m + y f - d = 0. \tag{9.45}
\]
In terms of the abbreviations \( \epsilon_{w\ell} = (\epsilon_w + \epsilon_\ell)/2 \) and \( \epsilon_{c\ell} = (\epsilon_c + \epsilon_\ell)/2 \) as well as \( p = (\epsilon_w - \epsilon_\ell)/(\epsilon_w + \epsilon_\ell) \) and \( p' = (\epsilon_c - \epsilon_\ell)/(\epsilon_c + \epsilon_\ell) \), the solutions are
\[
u(k) = \beta \frac{p - p'/y}{1 - pp'/y} \quad \text{and} \quad m(k) = \beta \frac{\epsilon_w}{\epsilon_{w\ell}} \frac{1}{1 - pp'/y} \]
\[f(k) = -\beta \frac{\epsilon_w}{\epsilon_{w\ell}} \frac{p'/y}{1 - pp'/y} \quad \text{and} \quad d(k) = \beta \frac{\epsilon_w\epsilon_\ell}{\epsilon_{c\ell}} \frac{1}{1 - pp'/y}. \tag{9.46}\]
Inserting these solutions into the Bessel expansions (9.42) for the potentials, expanding their denominators
\[
\frac{1}{1 - pp'/y} = \sum_{n=0}^{\infty} (pp')^n e^{-2nkt} \tag{9.47}\]
and using the integral (9.43), we find that the potential \( V_w \) in the extracellular water of a charge \( q \) at \((0, 0, h)\) in the water is
\[
V_w(p, z) = \frac{q}{4\pi \epsilon_w} \left[ \frac{1}{r} + \frac{p}{\sqrt{\rho^2 + (z + h)^2}} - \sum_{n=1}^{\infty} \frac{p' (1 - p^2) (pp')^{n-1}}{\sqrt{\rho^2 + (z + 2nt + h)^2}} \right] \tag{9.48}\]
in which \( r = \sqrt{\rho^2 + (z - h)^2} \) is the distance to the charge \( q \). The principal image charge \( pq \) is at \((0, 0, -h)\). Similarly, the potential \( V_e \) in the lipid bilayer is
\[
V_e(p, z) = \frac{q \epsilon_\ell}{4\pi \epsilon_w \epsilon_\ell e_c} \sum_{n=0}^{\infty} \left[ (pp')^n \frac{1}{\sqrt{\rho^2 + (z - 2nt - h)^2}} - \frac{p^n p'^{n+1}}{\sqrt{\rho^2 + (z + 2(n+1)t + h)^2}} \right] \tag{9.49}\]
and that in the cytosol is
\[
V_c(p, z) = \frac{q \epsilon_\ell}{4\pi \epsilon_w \epsilon_\ell e_c} \sum_{n=0}^{\infty} \frac{(pp')^n}{\sqrt{\rho^2 + (z - 2nt - h)^2}}. \tag{9.50}\]
These potentials are the same as those of example 4.16, but this derivation is much simpler and less error prone than the method of images.
Since \( p = (\epsilon_w - \epsilon_\ell)/(\epsilon_w + \epsilon_\ell) > 0 \), the principal image charge \( pq \) at \((0, 0, -h)\) has the same sign as the charge \( q \) and so contributes a positive term proportional to \( pq^2 \) to the energy. So a lipid membrane repels a nearby charge in water no matter what the sign of the charge. In a mean-field theory, all of the particles move under the influence of a common potential
\( V(x) \), and so the force \( qE(x) = -q\nabla V(x) \) points in *opposite* directions for charges of opposite signs. No mean-field theory can describe how a lipid slab repels both cations and anions that are nearby in water. \( \square \)

**Example 9.3 (Cylindrical Wave Guides)** An electromagnetic wave traveling in the \( z \)-direction down a cylindrical wave guide looks like

\[
E e^{i(kz - \omega t)} \quad \text{and} \quad B e^{i(kz - \omega t)}
\]

(9.51)
in which \( E \) and \( B \) depend upon \( \rho \) and \( \phi \)

\[
E = E_{\rho}\hat{\rho} + E_{\phi}\hat{\phi} + E_z\hat{z} \quad \text{and} \quad B = B_{\rho}\hat{\rho} + B_{\phi}\hat{\phi} + B_z\hat{z}
\]

(9.52)
in cylindrical coordinates (11.164–11.169 & 11.241). If the wave guide is an evacuated, perfectly conducting cylinder of radius \( r \), then on the surface of the wave guide the parallel components of \( E \) and the normal component of \( B \) must vanish which leads to the boundary conditions

\[
E_z(r, \phi) = 0, \quad E_{\phi}(r, \phi) = 0, \quad \text{and} \quad B_{\rho}(r, \phi) = 0.
\]

(9.53)

Since the \( E \) and \( B \) fields have subscripts, we will use commas to denote derivatives as in \( \partial E_z/\partial \phi \equiv E_{z,\phi} \) and \( \partial(\rho E_{\phi})/\partial \rho \equiv (\rho E_{\phi})_{,\rho} \) and so forth. In this notation, the vacuum forms \( \nabla \times E = -\dot{B} \) and \( \nabla \times B = \dot{E}/c^2 \) of the Faraday and Maxwell-Ampère laws give us (exercise 9.14) the field equations

\[
\begin{align*}
E_{z,\phi}/\rho - ikE_{\phi} &= i\omega B_{\rho} & \quad E_{z,\phi}/\rho - ikE_{\phi} &= -i\omega E_{\rho}/c^2 \\
ike E_{\rho} - E_{z,\phi} &= i\omega B_{\phi} & \quad ikB_{\rho} - B_{z,\phi} &= -i\omega E_{\phi}/c^2 \\
[(\rho E_{\phi})_{,\rho} - E_{\rho,\phi}] / \rho &= i\omega B_z & \quad [(\rho B_{\phi})_{,\phi} - B_{\phi,\rho}] / \rho &= -i\omega E_z/c^2.
\end{align*}
\]

(9.54)

Solving them for the \( \rho \) and \( \phi \) components of \( E \) and \( B \) in terms of their \( z \) components (exercise 9.15), we find

\[
\begin{align*}
E_{\rho} &= -i\frac{kE_{z,\rho} + \omega B_{z,\rho}}{k^2 - \omega^2/c^2} & \quad E_{\phi} &= -i\frac{kE_{z,\phi} / \rho - \omega B_{z,\phi}}{k^2 - \omega^2/c^2} \\
B_{\rho} &= -i\frac{kB_{z,\rho} - \omega E_{z,\rho}/c^2}{k^2 - \omega^2/c^2} & \quad B_{\phi} &= -i\frac{kB_{z,\phi} / \rho + \omega E_{z,\phi}/c^2}{k^2 - \omega^2/c^2}.
\end{align*}
\]

(9.55)

The fields \( E_z \) and \( B_z \) obey the separable wave equations (6.61, 11.91)

\[
-\Delta E_z = -\tilde{E}_z/c^2 = \omega^2 E_z/c^2 \quad \text{and} \quad -\Delta B_z = -\tilde{B}_z/c^2 = \omega^2 B_z/c^2.
\]

(9.56)

Because their \( z \)-dependence (9.51) is periodic, they are (exercise 9.16) linear combinations of \( J_n(\sqrt{\omega^2/c^2 - k^2}\rho)e^{in\phi}e^{i(kz - \omega t)} \).

Modes with \( B_z = 0 \) are **transverse magnetic** or **TM** modes. For them
the boundary conditions (9.53) will be satisfied if \( \sqrt{\omega^2/c^2 - k^2 r} \) is a zero \( z_{n,m} \) of \( J_n \). So the frequency \( \omega_{n,m}(k) \) of the \( n, m \) TM mode is

\[
\omega_{n,m}(k) = c \sqrt{k^2 + z_{n,m}^2/r^2},
\]

(9.57)

Since first zero of a Bessel function is \( z_{0,1} \approx 2.4048 \), the minimum frequency \( \omega_{0,1}(0) = c z_{0,1}/r \approx 2.4048 c/r \) occurs for \( n = 0 \) and \( k = 0 \). If the radius of the wave-guide is \( r = 1 \) cm, then \( \omega_{0,1}(0)/2\pi \) is about 11 GHz, which is a microwave frequency with a wavelength of 2.6 cm. In terms of the frequencies (9.57), the field of a pulse moving in the \( +z \)-direction is

\[
E_z(\rho, \phi, z, t) = \sum_{n=0}^{\infty} \sum_{m=1}^{\infty} \int_0^\infty c_{n,m}(k) J_n \left( \frac{z_{n,m} \rho}{r} \right) e^{im\phi} \exp i [k z - \omega_{n,m}(k)t] \, dk.
\]

(9.58)

Modes with \( E_z = 0 \) are **transverse electric** or TE modes. For them the boundary conditions (9.53) will be satisfied (exercise 9.18) if \( \sqrt{\omega^2/c^2 - k^2 r} \) is a zero \( z'_{n,m} \) of \( J'_n \). Their frequencies are \( \omega_{n,m}(k) = c \sqrt{k^2 + z'^2_{n,m}/r^2} \).

Since first zero of a first derivative of a Bessel function is \( z'_{1,1} \approx 1.8412 \), the minimum frequency \( \omega_{1,1}(0) = c z'_{1,1}/r \approx 1.8412 c/r \) occurs for \( n = 1 \) and \( k = 0 \). If the radius of the wave-guide is \( r = 1 \) cm, then \( \omega_{1,1}(0)/2\pi \) is about 8.8 GHz, which is a microwave frequency with a wavelength of 3.4 cm.

**Example 9.4 (Cylindrical Cavity)** The modes of an evacuated, perfectly conducting cylindrical cavity of radius \( r \) and height \( h \) are like those of a cylindrical wave guide (example 9.3) but with added boundary conditions

\[
B_z(\rho, \phi, 0, t) = 0 \quad \text{and} \quad B_z(\rho, \phi, h, t) = 0
\]

\[
E_\rho(\rho, \phi, 0, t) = 0 \quad \text{and} \quad E_\rho(\rho, \phi, h, t) = 0
\]

\[
E_\phi(\rho, \phi, 0, t) = 0 \quad \text{and} \quad E_\phi(\rho, \phi, h, t) = 0
\]

(9.59)

at the two ends of the cylinder. If \( \ell \) is an integer and if \( \sqrt{\omega^2/c^2 - \pi^2 \ell^2/h^2} r \) is a zero \( z'_{n,m} \) of \( J'_n \), then the TE fields \( E_z = 0 \) and

\[
B_z = J_n(z_{n,m} \rho/r) e^{im\phi} \sin(\pi \ell z/h) e^{-i\omega t}
\]

(9.60)

satisfy both these (9.59) boundary conditions at \( z = 0 \) and \( h \) and those (9.53) at \( \rho = r \) as well as the separable wave equations (9.56). The frequencies of the resonant TE modes then are \( \omega_{n,m,\ell} = c \sqrt{z'^2_{n,m}/r^2 + \pi^2 \ell^2/h^2} \).

The TM modes are \( B_z = 0 \) and

\[
E_z = J_n(z_{n,m} \rho/r) e^{im\phi} \cos(\pi \ell z/h) e^{-i\omega t}
\]

(9.61)

with resonant frequencies \( \omega_{n,m,\ell} = c \sqrt{z'^2_{n,m}/r^2 + \pi^2 \ell^2/h^2} \).
9.2 Spherical Bessel Functions

If in Bessel’s equation (9.3), one sets \( n = \ell + 1/2 \) and \( j_\ell = \sqrt{\pi/2x} J_{\ell+1/2} \), then one may show (exercise 9.21) that

\[
x^2 j''_\ell(x) + 2x j'_\ell(x) + [x^2 - \ell(\ell + 1)] j_\ell(x) = 0
\]

(9.62)

which is the equation for the spherical Bessel function \( j_\ell \).

We saw in example 6.8 that by setting \( V(r, \theta, \phi) = R_{k,\ell}(r) \Theta_{\ell,m}(\theta) \Phi_m(\phi) \) we could separate the variables of Helmholtz’s equation

\[
4V = k^2 V
\]

in spherical coordinates

\[
\frac{r^2 \Delta V}{V} = \frac{\left( r^2 R_{k,\ell}' \right)'}{R_{k,\ell}} + \frac{\left( \sin \theta \Theta_{\ell,m}' \right)'}{\sin \theta \Theta_{\ell,m}} + \frac{\Phi''}{\sin^2 \theta \Phi} = -k^2 r^2.
\]

(9.63)

Thus if \( \Phi_m(\phi) = e^{im\phi} \) so that \( \Phi''_m = -m^2 \Phi_m \), and if \( \Theta_{\ell,m} \) satisfies the associated Legendre equation

\[
sin \theta \left( \sin \theta \Theta'_{\ell,m} \right)' + \left[ \ell(\ell + 1) \sin^2 \theta - m^2 \right] \Theta_{\ell,m} = 0
\]

(9.64)

then the product \( V(r, \theta, \phi) = R_{k,\ell}(r) \Theta_{\ell,m}(\theta) \Phi_m(\phi) \) will obey (9.63) because in view of (9.62) the radial function \( R_{k,\ell}(r) = j_\ell(kr) \) satisfies

\[
\left( r^2 R_{k,\ell}' \right)' + [k^2 r^2 - \ell(\ell + 1)] R_{k,\ell} = 0.
\]

(9.65)

In terms of the spherical harmonic \( Y_{\ell,m}(\theta, \phi) = \Theta_{\ell,m}(\theta) \Phi_m(\phi) \), the solution is \( V(r, \theta, \phi) = j_\ell(kr) Y_{\ell,m}(\theta, \phi) \).

Rayleigh’s formula gives the spherical Bessel function

\[
j_\ell(x) \equiv \sqrt{\frac{\pi}{2x}} J_{\ell+1/2}(x)
\]

(9.66)
after the \( \ell \)th derivative of \( \sin x/x \)

\[
j_\ell(x) = (-1)\ell x^\ell \left( \frac{1}{x} \frac{d}{dx} \right) \ell \left( \frac{\sin x}{x} \right)
\]

(9.67)

(Lord Rayleigh (John William Strutt) 1842–1919). In particular, \( j_0(x) = \sin x/x \) and \( j_1(x) = \sin x/x^2 - \cos x/x \). Rayleigh’s formula leads to the recursion relation (exercise 9.22)

\[
j_{\ell+1}(x) = \frac{\ell}{x} j_\ell(x) - j'_\ell(x)
\]

(9.68)

with which one can show (exercise 9.23) that the spherical Bessel functions
as defined by Rayleigh’s formula do satisfy their differential equation (9.65) with \( x = kr \).

The spherical Bessel functions \( j_\ell (kr) \) satisfy the self-adjoint Sturm-Liouville (6.340) equation (9.65)

\[
- r^2 j''_\ell - 2r j'_\ell + \ell (\ell + 1) j_\ell = k^2 r^2 j_\ell
\]

(9.69)

with eigenvalue \( k^2 \) and weight function \( \rho = r^2 \). If \( j_\ell (z_{\ell,n}) = 0 \), then the functions \( j_\ell (kr) = j_\ell (z_{\ell,n} r/a) \) vanish at \( r = a \) and form an orthogonal basis

\[
\int_0^a j_\ell(z_{\ell,n} r/a) j_\ell(z_{\ell,m} r/a) r^2 \, dr = \frac{a^3}{2} j_{\ell+1}(z_{\ell,n}) \delta_{n,m}
\]

(9.70)

for a self-adjoint system on the interval \([0, a]\). Moreover, since the eigenvalues \( k_{\ell,n}^2 = z_{\ell,n}^2 / a^2 \approx (n\pi)^2 / a^2 \to \infty \) as \( n \to \infty \), the eigenfunctions \( j_\ell(z_{\ell,n} r/a) \) also are complete in the mean.

On an infinite interval, the analogous relation is

\[
\int_0^\infty j_\ell(k r) j_\ell(k' r) r^2 \, dr = \frac{\pi}{2k^2} \delta(k - k').
\]

(9.71)

If we write the spherical Bessel function \( j_0(x) \) as the integral

\[
j_0(z) = \frac{\sin z}{z} = \frac{1}{2} \int_{-1}^1 e^{ixz} \, dx
\]

(9.72)

and use Rayleigh’s formula (9.67), we may find an integral for \( j_\ell(z) \)

\[
j_\ell(z) = (-1)^\ell z^\ell \left( \frac{1}{z} \frac{d}{dz} \right)^\ell \left( \frac{\sin z}{z} \right) = (-1)^\ell z^\ell \left( \frac{1}{z} \frac{d}{dz} \right)^\ell \frac{1}{2} \int_{-1}^1 e^{ixz} \, dx
\]

\[
= \frac{z^\ell}{2} \int_{-1}^1 (1 - x^2)^\ell \, e^{ixz} \, dx = \frac{(-i)^\ell}{2} \int_{-1}^1 (1 - x^2)^\ell \, \frac{d^\ell}{dx^\ell} e^{ixz} \, dx
\]

\[
= \frac{(-i)^\ell}{2} \int_{-1}^1 e^{ixz} \, \frac{d^\ell}{dx^\ell} (x^2 - 1)^\ell \, dx = \frac{(-i)^\ell}{2} \int_{-1}^1 P_\ell(x) \, e^{ixz} \, dx
\]

(9.73)

(exercise 9.24) that contains Rodrigues’s formula (8.8) for the Legendre polynomial \( P_\ell(x) \). With \( z = kr \) and \( x = \cos \theta \), this formula

\[
i^\ell j_\ell(kr) = \frac{1}{2} \int_{-1}^1 P_\ell(\cos \theta) e^{ikr \cos \theta} \, d \cos \theta
\]

(9.74)
Bessel Functions

and the Fourier-Legendre expansion (8.32) gives

\[ e^{ikr \cos \theta} = \sum_{\ell=0}^{\infty} \frac{2\ell + 1}{2} P_\ell(\cos \theta) \int_{-1}^{1} P_\ell(\cos \theta') e^{ikr \cos \theta'} d \cos \theta' \]

\[ = \sum_{\ell=0}^{\infty} (2\ell + 1) P_\ell(\cos \theta) i^\ell j_\ell(kr). \]  

(9.75)

If \( \theta, \phi \) and \( \theta', \phi' \) are the polar angles of the vectors \( r \) and \( k \), then by using the addition theorem (8.122) we get

\[ e^{ik \cdot r} = \sum_{\ell=0}^{\infty} 4\pi i^\ell j_\ell(kr) Y_{\ell,m}(\theta, \phi) Y_{\ell,m}^*(\theta', \phi'). \]  

(9.76)

Example 9.5 (Plane Waves) The inner product of the bra \( \langle r \rangle \) that represents a particle at \( r \) with polar angles \( \theta \) and \( \phi \) and the ket \( | k \rangle \) that represents a particle with momentum \( p = \hbar k \) with polar angles \( \theta' \) and \( \phi' \) is

\[ \langle r | k \rangle = \frac{1}{(2\pi)^{3/2}} e^{ik r \cos \theta} = \frac{1}{(2\pi)^{3/2}} \sum_{\ell=0}^{\infty} (2\ell + 1) P_\ell(\cos \theta) i^\ell j_\ell(kr) \]

\[ = \frac{1}{(2\pi)^{3/2}} e^{ik \cdot r} = \sqrt{\frac{2}{\pi}} \sum_{\ell=0}^{\infty} i^\ell j_\ell(kr) Y_{\ell,m}(\theta, \phi) Y_{\ell,m}^*(\theta', \phi'). \]  

(9.77)

in which \( k \cdot r = kr \cos \theta. \)

The series expansion (9.1) for \( J_n \) and the definition (9.66) of \( j_\ell \) give us for small \( |p| \ll 1 \) the approximation

\[ j_\ell(p) \approx \frac{\ell! (2\rho)^\ell}{(2\ell + 1)!} = \frac{\rho^\ell}{(2\ell + 1)!!}. \]  

(9.78)

To see how \( j_\ell(p) \) behaves for large \( |p| \gg 1 \), we use Rayleigh’s formula (9.67) to compute \( j_1(p) \) and notice that the derivative \( d/d\rho \)

\[ j_1(p) = -\frac{d}{d\rho} \left( \frac{\sin \rho}{\rho} \right) = -\frac{\cos \rho}{\rho} + \frac{\sin \rho}{\rho^2}. \]  

(9.79)

adds a factor of \( 1/\rho \) when it acts on \( 1/\rho \) but not when it acts on \( \sin \rho \). Thus the dominant term is the one in which all the derivatives act on the sine, and so for large \( |\rho| \gg 1 \), we have approximately

\[ j_\ell(p) = (-1)^\ell \rho^\ell \left( \frac{1}{\rho} \frac{d}{d\rho} \right)^\ell \left( \frac{\sin \rho}{\rho} \right) = (-1)^\ell \rho^\ell \frac{\sin (\rho - \ell \pi / 2)}{\rho} \]  

with an error that falls off as \( 1/\rho^2 \). The quality of the approximation, which is exact for \( \ell = 0 \), is illustrated for \( \ell = 1 \) and 2 in Fig. 9.2.
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The Spherical Bessel Function $j_1(\rho)$ for $\rho \ll 1$ and $\rho \gg 1$

The Spherical Bessel Function $j_2(\rho)$ for $\rho \ll 1$ and $\rho \gg 1$

Figure 9.2 Top: Plot of $j_1(\rho)$ (solid curve) and its approximations $\rho/3$ for small $\rho$ (9.78, dashes) and $\sin(\rho - \pi/2)/\rho$ for big $\rho$ (9.80, dot-dash). Bottom: Plot of $j_2(\rho)$ (solid curve) and its approximations $\rho^2/15$ for small $\rho$ (9.78, dashed) and $\sin(\rho - \pi)/\rho$ for big $\rho$ (9.80, dot-dash). The values of $\rho$ at which $j_\ell(\rho) = 0$ are the zeros or roots of $j_\ell$; we use them to fit boundary conditions.

Example 9.6 (Partial Waves) Spherical Bessel functions occur in the wave-functions of free particles with well defined angular momentum.

The hamiltonian $H_0 = p^2/2m$ for a free particle of mass $m$ and the square $L^2$ of the orbital angular momentum operator are both invariant under rotations; thus they commute with the orbital angular momentum operator $L$. Since the operators $H_0$, $L^2$, and $L_z$ commute with each other, simultaneous eigenstates $|k, \ell, m\rangle$ of these compatible operators (section 1.30) exist

$$H_0 |k, \ell, m\rangle = \frac{p^2}{2m} |k, \ell, m\rangle = \frac{(\hbar k)^2}{2m} |k, \ell, m\rangle$$

$$L^2 |k, \ell, m\rangle = \hbar^2 \ell(\ell + 1) |k, \ell, m\rangle \quad \text{and} \quad L_z |k, \ell, m\rangle = \hbar m |k, \ell, m\rangle.$$  

By (9.62–9.65), their wave-functions are products of spherical Bessel func-
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\[ \langle r | k, \ell, m \rangle = \langle r, \theta, \phi | k, \ell, m \rangle = \sqrt{\frac{2}{\pi}} k j_\ell(kr) Y_{\ell,m}(\theta, \phi). \quad (9.82) \]

They satisfy the normalization condition

\[
\langle k, \ell, m | k, \ell', m' \rangle = \frac{2k k'}{\pi} \int_0^\infty j_\ell(kr) j_\ell(k'r) r^2 \, dr \int Y^*_{\ell,m}(\theta, \phi) Y_{\ell',m'}(\theta, \phi) \, d\Omega
\]

\[ = \delta(k - k') \delta_{\ell,\ell'} \delta_{m,m'} \quad (9.83) \]

and the completeness relation

\[
1 = \int_0^\infty \int_{-\infty}^{\infty} \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} |k, \ell, m \rangle \langle k, \ell, m|.
\]

Their inner products with an eigenstate \( |k'\rangle \) of a free particle of momentum \( p' = \hbar k' \) are

\[
\langle k, \ell, m | k', \ell', m' \rangle = \frac{i^\ell}{k} \delta(k - k') Y^*_{\ell,m}(\theta', \phi') \quad (9.85)
\]
in which the polar coordinates of \( k' \) are \( \theta', \phi' \).

Using the resolution (9.84) of the identity operator and the inner-product formulas (9.82 & 9.85), we recover the expansion (9.76)

\[
\frac{e^{ik' \cdot r}}{(2\pi)^{3/2}} = |r | k' \rangle = \int_0^\infty \int_{-\infty}^{\infty} \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \langle r | k, \ell, m \rangle \langle k, \ell, m | k' \rangle
\]

\[ = \sum_{\ell=0}^{\infty} \sqrt{\frac{2}{\pi}} i^\ell j_\ell(kr) Y^*_{\ell,m}(\theta, \phi) Y_{\ell,m}(\theta', \phi'). \quad (9.86) \]

The small \( kr \) approximation (9.78) and the definition (9.82) tell us that the probability that a particle with angular momentum \( \hbar \ell \) about the origin has \( r = |r| \ll 1/k \) is

\[
P(r) = \frac{2k^2}{\pi} \int_0^r j_\ell^2(kr') r'^2 \, dr' \approx \frac{2}{\pi[(2\ell + 1)!!]^2} \int_0^r (kr)^{2\ell+2} \, dr = \frac{(4\ell + 6)(kr)^{2\ell+3}}{\pi[(2\ell + 3)!!]^2 k}
\]

which is very small for big \( \ell \) and tiny \( k \). So a short-range potential can only affect partial waves of low angular momentum. When physicists found that nuclei scattered low-energy hadrons into \( s \)-waves, they knew that the range of the nuclear force was short, about \( 10^{-15} \text{m} \).

If the potential \( V(r) \) that scatters a particle is of short range, then at big
9.2 Spherical Bessel Functions

The radial wave-function \( u_\ell(r) \) of the scattered wave should look like that of a free particle (9.86) which by the big \( kr \) approximation (9.80) is

\[
u^{(0)}_\ell(r) = j_\ell(kr) \approx \sin(kr - l\pi/2) \quad \Rightarrow \quad \frac{1}{2ikr} \left[ e^{i(kr - l\pi/2)} - e^{-i(kr - l\pi/2)} \right].
\]

(9.88)

Thus at big \( r \) the radial wave-function \( u_\ell(r) \) differs from \( u^{(0)}_\ell(r) \) only by a phase shift \( \delta_\ell \)

\[
u_\ell(r) \approx \frac{\sin(kr - l\pi/2 + \delta_\ell)}{kr} = \frac{1}{2ikr} \left[ e^{i(kr - l\pi/2 + \delta_\ell)} - e^{-i(kr - l\pi/2 + \delta_\ell)} \right].
\]

(9.89)

The phase shifts determine the cross-section \( \sigma \) to be (Cohen-Tannoudji et al., 1977, chap. VIII)

\[
\sigma = \frac{4\pi}{k^2} \sum_{l=0}^{\infty} (2l + 1) \sin^2 \delta_\ell.
\]

(9.90)

If the potential \( V(r) \) is negligible for \( r > r_0 \), then for momenta \( k \ll 1/r_0 \) the cross-section is \( \sigma \approx 4\pi \sin^2 \delta_0/k^2 \). 


Example 9.7 (Quantum Dots) The active region of some quantum dots is a CdSe sphere whose radius \( a \) is less than 2 nm. Photons from a laser excite electron-hole pairs which fluoresce in nanoseconds.

I will model a quantum dot simply as an electron trapped in a sphere of radius \( a \). Its wave-function \( \psi(r, \theta, \phi) \) satisfies Schrödinger’s equation

\[-\frac{\hbar^2}{2m} \Delta \psi = E\psi \]

(9.91)

with the boundary condition \( \psi(a, \theta, \phi) = 0 \). With \( k^2 = 2mE/\hbar^2 = z_{\ell,n}^2/a^2 \), the unnormalized eigenfunctions are

\[
\psi_{n,\ell,m}(r, \theta, \phi) = j_\ell(z_{\ell,n}r/a) Y_{\ell,m}(\theta, \phi) \theta(a - r)
\]

(9.92)

in which the Heaviside function \( \theta(a - r) \) makes \( \psi \) vanish for \( r > a \), and \( \ell \) and \( m \) are integers with \(-\ell \leq m \leq \ell\) because \( \psi \) must be single valued for all angles \( \theta \) and \( \phi \).

The zeros \( z_{\ell,n} \) of \( j_\ell(x) \) fix the energy levels as \( E_{n,\ell,m} = (h z_{\ell,n}/a)^2/2m \). For \( j_0(x) = \sin x/x \), they are \( z_{0,n} = n\pi \). So \( E_{n,0,0} = (h n\pi/a)^2/2m \). If the coupling to a photon is via a term like \( \mathbf{p} \cdot \mathbf{A} \), then one expects \( \Delta \ell = 1 \). The energy gap from the \( n, \ell = 1 \) state to the \( n = 1, \ell = 0 \) ground state thus is

\[
\Delta E_n = E_{n,1,0} - E_{1,0,0} = (z_{1,n}^2 - \pi^2) \frac{\hbar^2}{2ma^2}.
\]

(9.93)
Inserting factors of $c^2$ and using $hc = 197$ eV nm, and $mc^2 = 0.511$ MeV, we find from the zero $z_{1,2} = 7.72525$ that $\Delta E_2 = 1.89 (\text{nm}/a)^2$ eV, which is red light if $a = 1$ nm. The next zero $z_{1,3} = 10.90412$ gives $\Delta E_3 = 4.14 (\text{nm}/a)^2$ eV, which is in the visible if $1.2 < a < 1.5$ nm. The Mathematica command
\[ \text{Do[Print[N[BesselJZero[1.5, k]]], \{k, 1, 5, 1\}] \] gives the first five zeros of $j_1(x)$ to 6 significant figures.

### 9.3 Bessel Functions of the Second Kind

In section 7.5 we derived integral representations (7.55 & 7.56) for the Hankel functions $H^{(1)}_{\lambda}(z)$ and $H^{(2)}_{\lambda}(z)$ for $\text{Re} z > 0$. One may analytically continue them (Courant and Hilbert, 1955, chap. VII) to the upper half $z$-plane
\[ H^{(1)}_{\lambda}(z) = \frac{1}{\pi i} e^{-i\lambda/2} \int_{-\infty}^{\infty} e^{iz\cosh x - \lambda x} \, dx \quad \text{Im} z \geq 0 \]
and to the lower half $z$-plane
\[ H^{(2)}_{\lambda}(z) = -\frac{1}{\pi i} e^{i\lambda/2} \int_{-\infty}^{\infty} e^{-iz\cosh x - \lambda x} \, dx \quad \text{Im} z \leq 0. \]

When both $z = \rho$ and $\lambda = \nu$ are real, the two Hankel functions are complex conjugates of each other
\[ H^{(1)}_{\nu}(\rho) = H^{(2)*}_{\nu}(\rho). \]

Hankel functions, called **Bessel functions of the third kind**, are linear combinations of Bessel functions of the first $J_{\lambda}(z)$ and **second Y_{\lambda}(z) kind**
\[ H^{(1)}_{\lambda}(z) = J_{\lambda}(z) + iY_{\lambda}(z) \]
\[ H^{(2)}_{\lambda}(z) = J_{\lambda}(z) - iY_{\lambda}(z). \]

Bessel functions of the second kind also are called **Neumann functions**; the symbols $Y_{\lambda}(z) = N_{\lambda}(z)$ refer to the same function. They are infinite at $z = 0$ as illustrated in Fig. 9.3.

When $z = ix$ is imaginary, we get the **modified Bessel functions**
\[ I_{\alpha}(x) = i^{-\alpha}J_{\alpha}(ix) = \sum_{m=0}^{\infty} \frac{1}{m! \Gamma(m + \alpha + 1)} \left( \frac{x}{2} \right)^{2m+\alpha} \]
\[ K_{\alpha}(x) = \frac{\pi}{2} i^{\alpha+1} H^{(1)}_{\alpha}(ix) = \int_{0}^{\infty} e^{-x \cosh t} \cosh \alpha t \, dt. \]
The Bessel Functions of the Second Kind $Y_0(\rho), Y_1(\rho), Y_2(\rho)$

The Bessel Functions of the Second Kind $Y_3(\rho), Y_4(\rho), Y_5(\rho)$

Figure 9.3 Top: $Y_0(\rho)$ (solid curve), $Y_1(\rho)$ (dot-dash), and $Y_2(\rho)$ (dashed) for $0 < \rho < 12$. Bottom: $Y_3(\rho)$ (solid curve), $Y_4(\rho)$ (dot-dash), and $Y_5(\rho)$ (dashed) for $2 < \rho < 14$. The points at which Bessel functions cross the $\rho$-axis are called zeros or roots; we use them to satisfy boundary conditions.

Some simple cases are

$$I_{1/2}(z) = \sqrt{\frac{2}{\pi z}} \cosh z, \quad I_{1/2}(z) = \sqrt{\frac{2}{\pi z}} \sinh z, \quad K_{1/2}(z) = \sqrt{\frac{\pi}{2z}} e^{-z}.$$  \hspace{1cm} (9.99)

When do we need to use these functions? If we are representing functions that are finite at the origin $\rho = 0$, then we don’t need them. But if the point $\rho = 0$ lies outside the region of interest or if the function we are representing is infinite at that point, then we do need the $Y_\nu(\rho)$’s.

Example 9.8 (Coaxial Wave Guides) An ideal coaxial wave guide is perfectly conducting for $\rho < r_0$ and $\rho > r$, and the waves occupy the region $r_0 < \rho < r$. Since points with $\rho = 0$ are not in the physical domain of the
problem, the electric field \( E(\rho, \phi) \exp(i(kz - \omega t)) \) is a linear combination of Bessel functions of the first and second kinds with

\[
E_z(\rho, \phi) = a J_n(\sqrt{\omega^2/c^2 - k^2} \rho) + b Y_n(\sqrt{\omega^2/c^2 - k^2} \rho)
\]

(9.100) in the notation of example 9.3. A similar equation represents the magnetic field \( B_z \). The fields \( E \) and \( B \) obey the equations and boundary conditions of example 9.3 as well as

\[
E_z(r_0, \phi) = 0, \quad E_\phi(r_0, \phi) = 0, \quad \text{and} \quad B_\rho(r_0, \phi) = 0
\]

(9.101) at \( \rho = r_0 \). In TM modes with \( B_z = 0 \), one may show (exercise 9.26) that the boundary conditions \( E_z(r_0, \phi) = 0 \) and \( E_z(r, \phi) = 0 \) can be satisfied if

\[
J_n(x) Y_n(v x) - J_n(v x) Y_n(x) = 0
\]

(9.102) in which \( v = r/r_0 \) and \( x = \sqrt{\omega^2/c^2 - k^2} r_0 \). One can use the Matlab code

```matlab
n = 0.; v = 10.;
f=@(x)besselj(n,x).*bessely(n,v*x)-besselj(n,v*x).*bessely(n,x)
x=linspace(0,5,1000);
figure
plot(x,f(x)) % we use the figure to guess at the roots
grid on
options=optimset('tolx',1e-9);
fzero(f,0.3) % we tell fzero to look near 0.3
fzero(f,0.7)
fzero(f,1)
```
to find that for \( n = 0 \) and \( v = 10 \), the first three solutions are \( x_{0,1} = 0.3314, x_{0,2} = 0.6858, \) and \( x_{0,3} = 1.0377 \). Setting \( n = 1 \) and adjusting the guesses in the code, one finds \( x_{1,1} = 0.3941, x_{1,2} = 0.7331, \) and \( x_{1,3} = 1.0748 \). The corresponding dispersion relations are \( \omega_{n,i}(k) = c \sqrt{k^2 + x_{n,i}^2/r_0^2} \).

### 9.4 Spherical Bessel Functions of the Second Kind

Spherical Bessel functions of the second kind are defined as

\[
y_\ell(\rho) = \sqrt{\frac{\pi}{2\rho}} Y_{\ell+1/2}(\rho)
\]

(9.103) and Rayleigh formulas express them as

\[
y_\ell(\rho) = (-1)^{\ell+1} \rho^{\ell} \left( \frac{d}{\rho d\rho} \right)^\ell \left( \frac{\cos \rho}{\rho} \right).
\]

(9.104)
The Spherical Bessel Function $y_1(\rho)$ for $\rho \ll 1$ and $\rho \gg 1$

The Spherical Bessel Function $y_2(\rho)$ for $\rho \ll 1$ and $\rho \gg 1$

Figure 9.4 Top: Plot of $y_1(\rho)$ (solid curve) and its approximations $-1/\rho^2$ for small $\rho$ (9.106, dot-dash) and $-\cos(\rho-\pi/2)/\rho$ for big $\rho$ (9.105, dashed).

Bottom: Plot of $y_2(\rho)$ (solid curve) and its approximations $-3/\rho^4$ for small $\rho$ (9.106, dot-dash) and $-\cos(\rho-\pi)/\rho$ for big $\rho$ (9.105, dashed). The values of $\rho$ at which $y_2(\rho) = 0$ are the zeros or roots of $y_2$; we use them to fit boundary conditions. All six plots run from $\rho = 1$ to $\rho = 12$.

The term in which all the derivatives act on the cosine dominates at big $\rho$

$$y_1(\rho) \approx (-1)^{\ell+1} \frac{1}{\rho} \frac{d^{\ell+1} \cos \rho}{d\rho^{\ell+1}} = - \cos (\rho - \ell \pi/2) / \rho.$$  \hspace{1cm} (9.105)

The second kind of spherical Bessel functions at small $\rho$ are approximately

$$y_2(\rho) \approx -(2\ell - 1)!/\rho^{\ell+1}.$$  \hspace{1cm} (9.106)

They all are infinite at $x = 0$ as illustrated in Fig. 9.4.

**Example 9.9 (Scattering off a Hard Sphere)** In the notation of example 9.6, the potential of a hard sphere of radius $r_0$ is $V(r) = \infty \theta(r_0 - r)$ in which $\theta(x) = (x + |x|)/2|x|$ is Heaviside’s function. Since the point $r = 0$ is
not in the physical region, the scattered wave function is a linear combination of spherical Bessel functions of the first and second kinds

\[ u_\ell(r) = c_\ell \, j_\ell(kr) + d_\ell \, y_\ell(kr). \tag{9.107} \]

The boundary condition \( u_\ell(kr_0) = 0 \) fixes the ratio \( v_\ell = d_\ell/c_\ell \) of the constants \( c_\ell \) and \( d_\ell \). Thus for \( \ell = 0 \), Rayleigh’s formulas (9.67 & 9.104) and the boundary condition say that \( kr_0 u_0(r_0) = c_0 \sin(kr_0) - d_0 \cos(kr_0) = 0 \) or \( d_0/c_0 = \tan kr_0 \). The s-wave then is \( u_0(kr) = c_0 \sin(kr - kr_0)/(kr \cos kr_0) \), which tells us that the phase shift is \( \delta_0(k) = -kr_0 \). By (9.90), the cross-section at low energy is \( \sigma \approx 4\pi r_0^2 \), or four times the classical value.

Similarly, one finds (exercise 9.27) that the p-wave phase shift is

\[ \delta_1(k) = \frac{kr_0 \cos kr_0 - \sin kr_0}{\cos kr_0 + kr_0 \sin kr_0}. \tag{9.108} \]

For \( kr_0 \ll 1 \), we have \( \delta_1(k) \approx -(kr_0)^3/6 \); more generally the \( \ell \)th phase shift \( \delta_\ell(k) \approx (kr_0)^{2\ell+1} \) for a potential of range \( r_0 \) at low energy \( k \ll 1/r_0 \).  

Further Reading

A great deal is known about Bessel functions. Students may find *Mathematical Methods for Physics and Engineering* (Riley et al., 2006) as well as the classics *A Treatise on the Theory of Bessel Functions* (Watson, 1995), *A Course of Modern Analysis* (Whittaker and Watson, 1927, chap. XVII), and *Methods of Mathematical Physics* (Courant and Hilbert, 1955) of special interest.

Exercises

9.1 Show that the series (9.1) for \( J_n(\rho) \) satisfies Bessel’s equation (9.3).
9.2 Show that the generating function \( \exp(z(u - 1/u)/2) \) for the Bessel functions is invariant under the substitution \( u \to -1/u \).
9.3 Use the invariance of \( \exp(z(u - 1/u)/2) \) under \( u \to -1/u \) to show that \( J_{-n}(z) = (-1)^n J_n(z) \).
9.4 By writing the generating function (9.4) as the product of the exponentials \( \exp(zu/2) \) and \( \exp(-z/2u) \), derive the expansion

\[
\exp \left[ \frac{z}{2} (u - u^{-1}) \right] = \sum_{m=-\infty}^{\infty} \sum_{n=0}^{\infty} \left( \frac{z}{2} \right)^{m+n} \frac{u^{m+n}}{(m+n)!} \left( -\frac{1}{2} \right)^n \frac{u^{-n}}{n!}. \tag{9.109}
\]
9.5 From this expansion (9.109) of the generating function (9.4), derive the power-series expansion (9.1) for \( J_n(z) \).

9.6 In the formula (9.4) for the generating function \( \exp(z(u - 1/u)/2) \), replace \( u \) by \( \exp i\theta \) and then derive the integral representation (9.5) for \( J_n(z) \).

9.7 From the general integral representation (9.5) for \( J_n(z) \), derive the two integral formulas (9.6) for \( J_0(z) \).

9.8 Show that the integral representations (9.5 & 9.6) imply that for any integer \( n \neq 0 \), \( J_n(0) = 0 \), while \( J_0(0) = 1 \).

9.9 By differentiating the generating function (9.4) with respect to \( u \) and identifying the coefficients of powers of \( u \), derive the recursion relation

\[ J_{n-1}(z) + J_{n+1}(z) = \frac{2n}{z} J_n(z). \tag{9.110} \]

9.10 By differentiating the generating function (9.4) with respect to \( z \) and identifying the coefficients of powers of \( u \), derive the recursion relation

\[ J_{n-1}(z) - J_{n+1}(z) = 2 J'_n(z). \tag{9.111} \]

9.11 Show that the change of variables \( \rho = ax \) turns (9.3) into the self-adjoint form of Bessel’s equation (9.10).

9.12 Letting \( y = J_n(ax) \), equation (9.10) is \((xy)' + (x a^2 - n^2/x)y = 0\). Multiply this equation by \( xy' \), integrate from 0 to \( b \), and so show that if \( ab = z_{n,m} \) and \( J_n(z_{n,m}) = 0 \), then

\[ 2 \int_0^b x J_n'(ax) \, dx = b^2 J_n^2(z_{n,m}) \tag{9.112} \]

which is the normalization condition (9.13).

9.13 Show that with \( \lambda \equiv z^2/r_d^2 \), the change of variables \( \rho = zr/r_d \) and \( u(r) = J_n(\rho) \) turns \((-ru')' + n^2 u/r = \lambda ru \) into (9.24).

9.14 Use the formula (11.241) for the curl in cylindrical coordinates and the vacuum forms \( \nabla \times \mathbf{E} = -\mathbf{B} \) and \( \nabla \times \mathbf{B} = \mathbf{E}/c^2 \) of the laws of Faraday and Maxwell-Ampère to derive the field equations (9.54).

9.15 Derive equations (9.55) from (9.54).

9.16 Show that \( J_n(\sqrt{\omega^2/c^2 - k^2} \rho) e^{im\phi} e^{i(kz-\omega t)} \) is a traveling-wave solution (9.51) of the wave equations (9.56).

9.17 Find expressions for the non-zero TM fields in terms of the formula (9.58) for \( E_z \).

9.18 Show that the field \( B_z = J_n(\sqrt{\omega^2/c^2 - k^2} \rho) e^{im\phi} e^{i(kz-\omega t)} \) will satisfy the boundary conditions (9.53) if \( \sqrt{\omega^2/c^2 - k^2} r \) is a zero \( z'_{n,m} \) of \( J_n' \).
9.19 Show that if \( \ell \) is an integer and if \( \sqrt{\omega^2/c^2 - \pi^2/\hbar^2} r \) is a zero \( z'_{n,m} \) of \( J'_n \), then the fields \( E_z = 0 \) and \( B_z = J_n(z'_{n,m}\rho/r)\right) e^{im\phi} \sin(\ell\pi z/h) e^{-i\omega t} \) satisfy both the boundary conditions (9.53) at \( \rho = r \) and those (9.59) at \( z = 0 \) and \( h \) as well as the wave equations (9.56). Hint: Use Maxwell’s equations \( \nabla \times E = -\dot{B} \) and \( \nabla \times B = \dot{E}/c^2 \) as in (9.54).

9.20 Show that the resonant frequencies of the TM modes of the cavity of example 9.4 are \( \omega_{n,m,\ell} = c\sqrt{z'^2_{n,m}/r^2 + \pi^2\ell^2/h^2} \).

9.21 By setting \( n = \ell + 1/2 \) and \( j_{\ell} = \sqrt{\pi/2}\pi J_{\ell+1/2} \), show that Bessel’s equation (9.3) implies that the spherical Bessel function \( j'_{\ell} \) satisfies (9.62).

9.22 Show that Rayleigh’s formula (9.67) implies the recursion relation (9.68).

9.23 Use the recursion relation (9.68) to show by induction that the spherical Bessel functions \( j_{\ell}(x) \) as given by Rayleigh’s formula (9.67) satisfy their differential equation (9.65), which with \( x = kr \) is

\[-x^2 j''_{\ell} - 2x j'_{\ell} + \ell(\ell + 1) j_{\ell} = x^2 j_{\ell}.
\]

(9.113)

Hint: start by showing that \( j_0(x) = \sin(x)/x \) satisfies (9.65). This problem involves some tedium.

9.24 Iterate the trick

\[
\frac{d}{dz} \int_{-1}^{1} e^{izx} dx = \frac{i}{z} \int_{-1}^{1} xe^{izx} dx = \frac{i}{2z} \int_{-1}^{1} e^{izx} d(x^2 - 1)
\]

\[
= -\frac{i}{2z} \int_{-1}^{1} (x^2 - 1) e^{izx} dx = \frac{1}{2} \int_{-1}^{1} (x^2 - 1) e^{izx} dx
\]

(9.114)

to show that (Schwinger et al., 1998, p. 227)

\[
\left( \frac{d}{dz} \right)^\ell \int_{-1}^{1} e^{izx} dx = \int_{-1}^{1} \frac{(x^2 - 1)^\ell}{2^\ell \ell!} e^{izx} dx.
\]

(9.115)

9.25 Show that \((-1)^\ell \delta \sin \rho/d\rho \right) = \sin(\rho - \pi\ell/2) \) and so complete the derivation of the approximation (9.80) for \( j_{\ell}(\rho) \) for big \( \rho \).

9.26 In the context of examples 9.3 and 9.8, show that the boundary conditions \( E_z(r_0, \phi) = 0 \) and \( E_z(r, \phi) = 0 \) imply (9.102).

9.27 Show that for scattering off a hard sphere of radius \( r_0 \) as in example 9.9, the \( p \)-wave phase shift is given by (9.108).